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PARAMETRIC RESONANSE IN STATICALLY INDETERMINATE FRAMES

The technique for development and reduction of discrete dynamics models of frames is presented. Construction of the models is carried out using finite element method, generalized coordinates and tools of modern computer software. Parametric resonance in a statically indeterminate frame caused by external excitations is investigated. Main instability domains of the frame are determined.

Key words: parametric excitation, reduced model, buckling eigenmodes, dynamic instability domain.

The problem under consideration is parametric resonance in statically indeterminate frames caused by external influences. To obtain equations describing parametric oscillations of frames one can use equilibrium equation for static stability problems taking into consideration inertial forces and some components of frame unexcited state which may depend on time. It is believed that the ratio between the excitation frequency and the lowest eigenfrequency of the frame in the unexcited movement allows to apply quasi-static approximation and to neglect some displacements while determining the unexcited stress-strain state of the frame.

Operator equation which describes parametric oscillations of statically indeterminate frame [1] is

$$\tilde{M}\ddot{\upsilon}(t)+\left[\tilde{K}-\left(\alpha+\beta f(t)\right)\tilde{K}_G\right]\upsilon(t)=0\,, \qquad (1)$$

where $\tilde{M}$, $\tilde{K}$ are inertial and elastic operators, $\tilde{K}_G$ is component of operator of parametric forces presented in the equation of quasi-static equilibrium. Domain of the solutions $\upsilon(\bar{x},t)$ of equation (1) coincides with the domain of definition of the operator $\tilde{K}$. Operators $\tilde{M}$, $\tilde{K}$, $\tilde{K}_G$ are positive definite. Equation (1) is written for the case when the parametric force are specified up to two factors, one of which α describes the static component of external influence while the second $\beta f(t)$ corresponds to a component that varies in time.

For numerical computation the transition from the operator equation (1) to discrete dynamic model is carried out using the finite element method. Discrete dynamic model is written as ordinary differential equations

$$M\ddot{\bar{u}}(t)+\left[K-(\alpha+\beta f(t))K_G\right]\bar{u}(t)=0\,, \qquad (2)$$

where $\bar{u}(t)=(u_1(t),u_2(t),...,u_n(t))^T$ is the vector of nodal displacements, M, K, K_G are matrix mass, stiffness matrix and geometric stiffness matrix respectively. Reducing of the discrete model (2) is performed with the help of the method of generalized coordinates taking into account the characteristics of parametric excitation.

Nontrivial solution of system (2) can be approximated by the expression

$$\bar{u}(t)=V\bar{y}(t), \qquad (3)$$

where the $n\times m$ – matrix $V=(\bar{v}_1,\bar{v}_2,...,\bar{v}_m)$ is determined by the system of basis vectors $\{\bar{v}_i\}_{i=1}^m$ and $\bar{y}(t)=(y_1(t),y_2(t),...,y_m(t))^T$ is vector of generalized coordinates. Substituting (3) in the (2) we obtain the system of m ordinary differential equations in terms of the vector $\bar{y}(t)=(y_1(t),y_2(t),...,y_m(t))^T$ components

$$V^TMV\ddot{\bar{y}}(t)+\left[V^TKV-(\alpha+\beta f(t))V^TK_GV\right]\bar{y}(t)=0\,. \qquad (4)$$

System (4) may be written as

$$M^*\ddot{\bar{y}}(t)+\left[K^*-(\alpha+\beta f(t))K_G^*\right]\bar{y}(t)=0\,, \qquad (5)$$

where reduced mass matrix M^*, stiffness matrix K^* and geometric stiffness matrix K_G^* of dimension $m\times m$ are given by

$$M^*=V^TMV\,,\ K^*=V^TKV\,,\ K_G^*=V^TK_GV\,.$$

We accept m to be much less than n $(m<<n)$. The adequacy of the model (5) is verified by examining internal convergence with increase of m or by use of other basis vectors.

In this paper the development of the reduced model (5) runs using finite element analysis program NASTRAN [2]. Since this program, as well as other standard programs do not contain procedures for determining the reduced matrices M^*,K^*,K_G^*, one can take advantage of a procedure for estimating the system response to a given field of displacement $\bar{v}$, i.e. the procedure for computing the vector $K\bar{v}$, where K is stiffness matrix of a structure in a whole. The vectors $M\bar{v}$, $K_G\bar{v}$ can be determined applying such a procedure. Also the procedures for solving the inverse problem of static equilibrium as well as procedures for modal and buckling analysis are used.

The governing matrix equation for modal analysis of a structure is

$$\left(K-\omega^2 M\right)\bar{\varphi}=0\,. \tag{6}$$

Let ω_k $\left(k=1,2,...,m_1\right)$ be a vector of the discrete model eigenfrequencies and $\bar{\varphi}_k=\left(\bar{\varphi}_{1k},\,\bar{\varphi}_{2\mathrm{k}},\,...\,,\,\bar{\varphi}_{\mathrm{nk}}\right)^T$ $\left(k=1,2,...,m_1\right)$ be a set of its eigenvectors. Then the orthogonality conditions of the eigenvectors $\left\{\bar{\varphi}_i\right\}_{\mathrm{i=1}}^{\mathrm{m_1}}$ are

$$\begin{aligned}\bar{\varphi}_j{}^T M\bar{\varphi}_i&=0,\\ \bar{\varphi}_j{}^T K\bar{\varphi}_i&=0.\end{aligned}\qquad \left(i\neq j\right).$$

Using a subset of the eigenvectors $\left\{\bar{\varphi}_i\right\}_{\mathrm{i=1}}^{\mathrm{m_1}}$ the approximated displacement field may be expressed as

$$\bar{v}\cong\sum_{i=1}^{m_1}a_i\bar{\varphi}_i. \tag{7}$$

where

$$a_i=\frac{\bar{\varphi}_i^T K\bar{v}}{\bar{\varphi}_i^T K\bar{\varphi}_i}\,. \tag{8}$$

Using (6) one may obtain

$$M\bar{\varphi}_i=\frac{1}{\omega_i^2}K\bar{\varphi}_i. \tag{9}$$

Furthermore, left multiplying both side of (7) by mass matrix M and taking into consideration (9) we get

$$M\bar{v}\cong\sum_{i=1}^{m_1}a_i M\bar{\varphi}_i\cong\sum_{i=1}^{m_1}\frac{a_i}{\omega_i^2}K\bar{\varphi}_i. \tag{10}$$

The procedure for solution of static buckling problem may be used for computing the vector $K_G\bar{v}$. The problem of frame stability may be expressed as

$$\left(K+\lambda K_G\right)\bar{\psi}=0. \tag{11}$$

Let λ_k $\left(k=1,2,...,m_2\right)$ be critical values of (11), $\bar{\psi}_k=\left(\bar{\psi}_1,\,\bar{\psi}_2,\,...,\,\bar{\psi}_{\mathrm{m}_2}\right)$ is a set of buckling eigenvectors. Due to assumptions made above about properties of the matrices K and K_G these vectors are orthogonal

$$\begin{aligned}\bar{\psi}_j{}^T K_G\bar{\psi}_i&=0,\\ \bar{\psi}_j{}^T K\bar{\psi}_i&=0.\end{aligned}\qquad (i\neq j)\,. \tag{12}$$

Using a subset of buckling eigenvectors $\{\overline{\psi}_i\}_{i=1}^{m_2}$ displacement field $\overline{v}$ can be expressed approximately as

$$\overline{v} \cong \sum_{i=1}^{m_2} b_i \overline{\psi}_i . \tag{13}$$

Similarly to (10) one can write

$$K_G \overline{v} \cong \sum_{i=1}^{m_2} -\frac{b_i}{\lambda_i} K \overline{\psi}_i , \tag{14}$$

where

$$b_i = \frac{\overline{\psi}_i^T K \overline{v}}{\overline{\psi}_i^T K \overline{\psi}_i}. \tag{15}$$

Thus the problem of determining the vector $K_G \overline{v}$ can be regarded as solved.

Next we can introduce subset of vectors $\{\overline{\varphi}_i\}_{i=1}^{m_1}$ and $\{\overline{\psi}_i\}_{i=1}^{m_2}$ of the same dimension m, i.e. $m_1 = m_2 = m$. Using the above equations one can write

$$MV = K\Phi\Omega^{-1}\Phi^T KV, \tag{16}$$

$$K_G V = K\Psi\Lambda^{-1}\Psi^T KV, \tag{17}$$

where the matrices $\Phi = (\overline{\varphi}_1, \overline{\varphi}_2, \dots, \overline{\varphi}_m)$ and $\Psi = (\overline{\psi}_1, \overline{\psi}_2, \dots, \overline{\psi}_m)$ have dimension $n \times m$, whereas matrices $\Omega^{-1} = diag\left(\omega_1^{-2}, \omega_2^{-2}, \dots, \omega_m^{-2}\right)$ and $\Lambda^{-1} = diag\left(\lambda_1^{-2}, \lambda_2^{-2}, \dots, \lambda_m^{-2}\right)$ are diagonal.

Reduced mass matrix M^*, stiffness matrix K^* and geometric stiffness matrix K_G^* are calculated using relations

$$M^* = V^T K\Phi\Omega^{-1}\Phi^T KV, \tag{18}$$

$$K^* = V^T KV, \tag{19}$$

$$K_G^* = V^T K\Psi\Lambda^{-1}\Psi^T KV. \tag{20}$$

This paper mainly deals with the reduction of equations which describe statically indeterminate frame parametric oscillation excited by the influence of vertical axial loads. Geometrical and mechanical characteristics are taken as follows

− for the columns

$F = 0.007569\,\text{m}^2$, $I_z = I_y = 4.7741 \times 10^{-6}\ \text{m}^4$, $J = 8.0617 \times 10^{-6}\ \text{m}^4$,

$E = 2.06 \times 10^{11}\ \text{Pa}$, $\eta = 0.3$, $\rho = 7800\ \text{kg/m}^3$;

− for the horizontal beams

$F=0.007569\ \text{m}^2$, $I_z=I_y=4.7741\times10^{-6}\ \text{m}^4$, $J=8.0617\times10^{-6}\ \text{m}^4$,

$E=1.0\times10^{12}\ \text{Pa}$, $\eta=0.3$, $\rho=100000\ \text{kg/m}^3$.

Finite element model of the frame is shown in Figure 1. Model contains eight nodes and nine spatial beam elements with six degrees of freedom at each node. The frame is rigidly attached to the surface. Displacements along the Z axis and rotation around the X axis are restricted at others nodes.

Parametric oscillation of the frame with rigid horizontal beams can be described with the reduced system of equations (5) using only one spatial variable - linear horizontal displacement along the X axis.

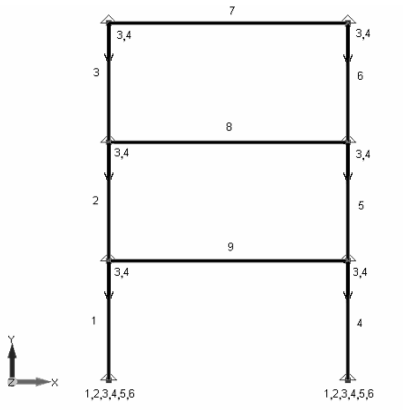


Figure 1. Finite-element model of the frame

$$M^*\ddot{\bar{x}}(t)+\left[K^*-\left(\alpha+\beta f(t)\right)K_G^*\right]\bar{x}(t)=0, \tag{21}$$

where $\bar{x}(t)=\{x_1(t),x_2(t),x_3(t)\}^T$ is a displacement vector. The parametric excitation is given by $P(t)=P_0+P_t=\alpha+\beta\cos\theta t$ and θ is the excitation frequency.

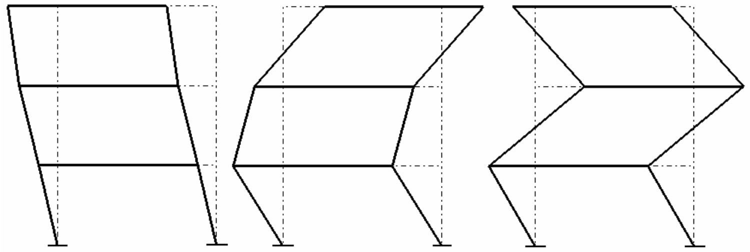
The tools of the FEM program NASTRAN is used to compute the reduced mass matrix, stiffness matrix and geometric stiffness matrix.

The generalized eigenvalue problem (6) was solved using Lanczos's method. Hereinafter only the first three eigenmodes shown in Figure 2 are taken into account to reduce the computation. Note that all the three eigenmodes are antisymmetric.

The numerical values of the eigenvectors elements are given below.

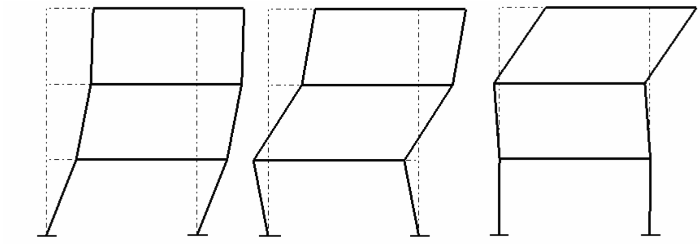
$$\{\bar{\varphi}_{n1}\}=\begin{Bmatrix}0\\-0,004103\\-0,008458\\-0,010966\\0\\-0,004103\\-0,008458\\-0,010966\end{Bmatrix},\quad \{\bar{\varphi}_{n2}\}=\begin{Bmatrix}0\\-0,010091\\-0,005731\\+0,008502\\0\\-0,010091\\-0,005731\\+0,008502\end{Bmatrix},\quad \{\bar{\varphi}_{n3}\}=\begin{Bmatrix}0\\+0,009256\\-0,009994\\+0,004405\\0\\+0,009256\\-0,009994\\+0,004405\end{Bmatrix}. \tag{22}$$

Buckling analysis of the frame loaded by axial forces (Figure 1) was performed in accordance with equation (11) using Lanczos's method. Critical loads $\lambda_1, \lambda_2, \lambda_3$ and correspondent buckling modes $\bar{\psi}_1, \bar{\psi}_2, \bar{\psi}_3$ are depicted in Figure 3.



$\omega_1 = 4{,}849\ s^{-1}$ $\omega_2 = 14{,}612\ s^{-1}$ $\omega_3 = 23{,}146\ s^{-1}$

Figure 2. Eigenmodes and eigenfrequencies of the frame



$\lambda_1 = 306729{,}8$ N $\lambda_2 = 438837{,}7$ N $\lambda_3 = 859537{,}6$ N

Figure 3. Buckling eigenmodes and corresponding critical values

The numerical values of the buckling eigenmodes are as follows

$$\{\bar{\psi}_{n1}\} = \begin{Bmatrix} 0 \\ +0{,}636694 \\ +0{,}949051 \\ +1{,}0 \\ 0 \\ +0{,}636694 \\ +0{,}949051 \\ +1{,}0 \end{Bmatrix}, \quad \{\bar{\psi}_{n2}\} = \begin{Bmatrix} 0 \\ -0{,}077949 \\ -0{,}022808 \\ -0{,}002823 \\ 0 \\ -0{,}077949 \\ -0{,}022808 \\ -0{,}002823 \end{Bmatrix}, \quad \{\bar{\psi}_{n3}\} = \begin{Bmatrix} 0 \\ -0{,}311108 \\ +0{,}719202 \\ +1{,}0 \\ 0 \\ -0{,}311108 \\ +0{,}719202 \\ +1{,}0 \end{Bmatrix}. \qquad (23)$$

Note that buckling eigenmodes are antisymmetric as well.

The reduced mass matrix is calculated using equations (18) and is approximately the unit matrix

$$M^{*} = diag[1{,}008058;\ 1{,}005287;\ 1{,}001766], \qquad (24)$$

which indicates that eigenvectors are normalized with respect to the mass matrix.

Solving the inverse static problems one can determine the structure response $K\overline{\varphi}_1$, $K\overline{\varphi}_2$, $K\overline{\varphi}_3$ to given field of displacements in the form of modal eigenvectors as well as response $K\overline{\psi}_1$, $K\overline{\psi}_2$, $K\overline{\psi}_3$ to displacements in the form of buckling eigenmodes.

The reduce stiffness matrix K^{*} is computed in accordance with (19):

$$K^{*} = \begin{bmatrix} 23{,}5954 & 0 & 0 \\ 0 & 213{,}6121 & 0 \\ 0 & 0 & 535{,}5765 \end{bmatrix}. \qquad (25)$$

The matrices $\Phi^{T} K \Psi$ and $\Psi^{T} K \Phi$ were obtained on purpose to compute the geometric stiffness matrix K_G^{*} accordingly to equation (20):

$$\Phi^{T} K \Psi = \begin{bmatrix} -2504{,}164 & 0 & 0 \\ 0 & 7703{,}229 & 0 \\ 0 & 0 & -14189{,}69 \end{bmatrix}, \qquad (26)$$

$$\Psi^{T} K \Phi = \begin{bmatrix} -2443{,}251 & 0 & 0 \\ 0 & 7548{,}467 & 0 \\ 0 & 0 & -14163{,}09 \end{bmatrix}. \qquad (27)$$

The reduced geometric stiffness matrix is

$$K_G^{*} = \begin{bmatrix} 0{,}66\times10^{-4} & 0 & 0 \\ 0 & 3{,}34\times10^{-4} & 0 \\ 0 & 0 & 3{,}59\times10^{-4} \end{bmatrix}. \qquad (28)$$

The reduced dynamic model of parametric oscillations of statically indeterminate frame is written as a system of three uncoupled second order differential equations

$$\begin{cases} \ddot{x}_1(t) + 4{,}858^2\left[1-0{,}279(\alpha+\beta\cos\theta t)\times10^{-5}\right]x_1(t)=0, \\ \ddot{x}_2(t)+14{,}615^2\left[1-0{,}156(\alpha+\beta\cos\theta t)\times10^{-5}\right]x_2(t)=0, \\ \ddot{x}_3(t)+23{,}143^2\left[1-0{,}067(\alpha+\beta\cos\theta t)\times10^{-5}\right]x_3(t)=0. \end{cases} \qquad (29)$$

Instability domains are areas in which any initial deviation increases indefinitely with time, i.e. undeformed shape of the frame is dynamically unstable. To determine the boundaries of the main domains of our frame instability the Bolotin's equation [1] can be applied

$$\left[K^* - \left(\alpha \pm \frac{1}{2}\beta\right)K_G^* - \frac{1}{4}\theta^2 M^*\right] = 0. \qquad (30)$$

The parametric oscillations become unstable when periodic or almost periodic solutions with periods 2T of the differential equations (29) exists. It is known that main resonances occur at frequencies of external load twice the eigenfrequencies of frame loaded by axial loads [3, 4].

Let us consider the dynamic stability of the frame under parametric load with static component $\alpha = [0;\ 0{,}25;\ 0{,}5]\lambda$ and dynamic component $\beta = [0{,}001 \div 0{,}5]\lambda$. The main instability domains near the first three eigenfrequencies are shown in Figure 4.

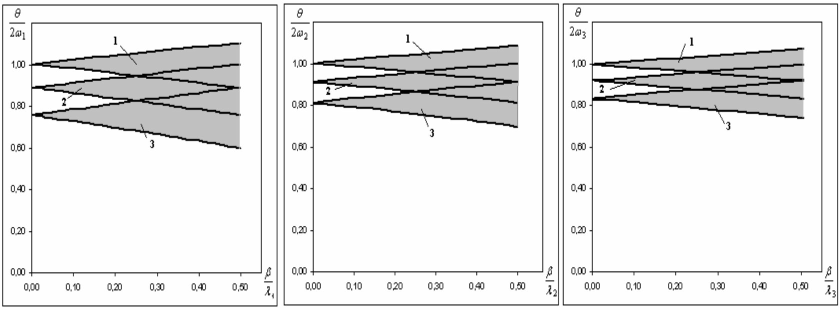


Figure 4. Main instability domains

The instability tongues in this figure are shaded. Digits 1, 2, 3 indicate the areas defined for the parameter α value equal to 0, 0,25 and 0,5 respectively. One can see that if the static component of parametric excitation is absent $(\alpha = 0)$ then the instability region starts with a frequency of disturbing force that is twice the unloaded frame eigenfrequency. When the static component has nonzero value then the parametric resonance begins with the frequency of disturbing force which is twice the eigenfrequency of the frame loaded with axial loads. The width of the main instability domains decreases with increasing of eigenfrequencies. Contrariwise the width of the resonance tongues increases slightly when the static component takes larger values.

REFERENCES

1. *Болотин В.В.* Динамическая устойчивость упругих систем. – М.: Гостехиздат, 1956. – 600 с.
2. *Шимкович Д.Г.* Расчет конструкций в MSC/NASTRAN for Windows. – М.: ДМК Пресс, 2001. – 448 с.
3. *Якубович В.А., Старжинский В.М.* Линейные дифференциальные уравнения с периодическими коэффициентами и их приложения. – М.: Наука, 1972. – 718 с.
4. Вибрации в технике: Справочник. В 6-ти т./Ред. совет: В.Н. Челомей (пред.). – М.: Машиностроение, 1978. – Т. 1. Колебания линейных систем /Под ред. В.В. Болотина. 1978. – 352 с.

1. *Bolotin V.V.* Dinamicheskaya ustoychivost' uprugikh sistem (The dynamic stability of elastic systems). – M.: Gostekhizdat, 1956. – 600 s.
2. *Shimkovich D.H.* Raschet konstruktsiy v MSC/NASTRAN for Windows (Structural analysis using MSC/NASTRAN for Windows). – M.: DMK Press, 2001. – 448 s.
3. *Yakubovich* V.A., Starzhinskiy V.M. Lineynye differentsial'nye uravneniya s periodicheskimi koeffitsientami i ikh prilozheniya (Linear differential equations with periodic coefficients and its applications). – M.: Nauka, 1972. – 718 s.
4. Vibratsii v tekhnyke: Spravochnik (Vibration in engineering: Handbook). V 6-ty t./Red. sovet: V.N. Chelomey (pred.). – M.: Mashynostroenye, 1978-T. 1. Kolebaniya lineynykh sistem (Oscillations of linear systems) /Pod red. V.V. Bolotina. 1978. – 352 s., il.

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ПАРАМЕТРИЧНИЙ РЕЗОНАНС В СТАТИЧНО НЕВИЗНАЧУВАНИХ РАМАХ

Представлено методику побудови редукованих дискретних динамічних моделей рам із застосуванням методу скінченних елементів, узагальнених координат і процедур сучасного програмного забезпечення. Досліджена стійкість параметричних коливань статично невизначуваної рами, спричинених зовнішнім впливом. Визначені головні області динамічної нестійкості рами.

Ключові слова: параметричне збудження, редукована модель, форми втрати стійкості, область динамічної нестійкості.

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ПАРАМЕТРИЧЕСКИЙ РЕЗОНАНС В СТАТИЧЕСКИ НЕОПРЕДЕЛИМЫХ РАМАХ

Представлена методика построения редуцированных дискретных динамических моделей рам с применением метода конечных элементов, обобщенных координат и процедур современного программного обеспечения. Исследована устойчивость параметрических колебаний статически неопределимой рамы, вызванных внешним возбуждением. Определены главные области динамической неустойчивости рамы.

Ключевые слова: параметрическое возбуждение, редуцированная модель, формы потери устойчивости, область динамической неустойчивости.