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# ANALYSIS OF ULTIMATE LIMIT LOAD OF SPHERICAL SHELLS APPLYING VON MISES YIELD CRITERION

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In this paper, the problems of ultimate limit load of spherical shell are formulated and solved. An equilibrium finite element developed by the method of Bubnov-Galerkin is suggested. Equilibrium and geometrical equations are created for this element and, based on these equations, the mathematical models of ultimate limit external load optimization problems for the shell structures are constructed. These are nonlinear mathematical programming problems. The methodology is illustrated by the numerical examples. The solution results are obtained for the finite elements of various sizes and show very high accuracy of the suggested element and convergence of the results.

## 1. Introduction

In this paper, the analysis of the stress-strain elastic-plastic states of a spherical shell subjected to symmetrical loading and the limit external load optimization are investigated. In the analysis problems, it is sought to define the internal forces, displacements and maximum external loading, when the geometry and physical parameters of the spherical shell material are known.

The spherical shell with a middle surface obtained as a circular curve revolution product in respect of a vertical axis is a separate case of the general shell [1–2]. The problem of the limit load optimization is presented in the static formulation, where unknowns are the static values – the generalized internal forces and the load parameter. However, the kinematical values can also be found by applying the duality theory. The physical model of the ideal elastic-plastic isotropic body and the Von Mises yield criteria are used [3–5]. Therefore, this problem is a nonlinear convex mathematical programming problem, where the equilibrium finite element method is used for discretization.

## 2. Discretization of spherical shell equations

The schematic view of the spherical shell is given in Fig 1. According to Kirchhoff-Love's theory, the shearing stress and strains are ignored. It means that the shear internal forces can be ignored also. For this reason, the stress state of the shell is defined by 6-dimensional vector of the internal forces
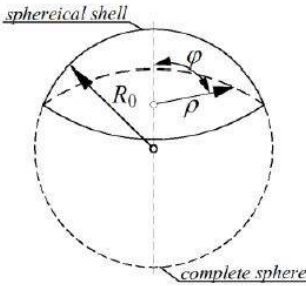
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Fig. 1. The basic notation for spherical shell

$$S = (M_{11}, M_{22}, M_{12}, N_{11}, N_{22}, N_{12})^T. \quad (1)$$

The spherical shell convenient to use in the polar coordinate system ($\rho$, $\varphi$, $z$). So the vector of the internal forces can be rewrite:

$$S = (M_\rho, M_\varphi, M_{\rho\varphi}, N_\rho, N_\varphi, N_{\rho\varphi})^T. \quad (2)$$

The six internal forces (1) or (2) are due to the so-called complete normal stress state of a shell. The positive direction of internal forces of the spherical shell are shown in Fig 2.

In the general case, when the load is non-symmetrically, the differential operator $\mathscr{H}$ of equilibrium equations of spherical shell is presented in Table 1.

Table 1

The operator of equilibrium equations on unsymmetrical load case

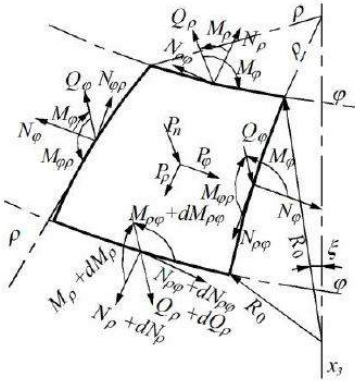| $M_\rho$ | $M_\varphi$ | $M_{\rho\varphi}$ | $N_\rho$ | $N_\varphi$ | $N_{\rho\varphi}$ |
|---|---|---|---|---|---|
|  |  |  | $-\frac{1}{\rho}\frac{\partial}{\partial\rho}$ | $\frac{1}{\rho}$ | $-\frac{1}{\rho}\frac{\partial}{\partial\rho}$ |
|  |  |  |  |  |  |
|  |  |  |  | $-\frac{1}{\rho}\frac{\partial}{\partial\phi}$ | $-\frac{2}{\rho}-\frac{\partial}{\partial\rho}$ |
| $-\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(1+\rho\frac{\partial}{\partial\rho}\right)$ | $\frac{1}{\rho}\frac{\partial}{\partial\rho}-\frac{1}{\rho^2}\frac{\partial^2}{\partial\rho^2}$ | $-\frac{2}{\rho^2}\frac{\partial}{\partial\phi}\left(1+\rho\frac{\partial}{\partial\rho}\right)$ | $-\frac{1}{R_0}$ | $-\frac{1}{R_0}$ |  |



Fig. 2. The positive direction of internal forces of the shell

In the case of a certain shell type design the number of internal forces can be reduced on the basis of introduced simplifying assumptions. If the load can be applied only symmetrically, then the duality condition requires the displacement to be symmetric too. Consequently, one can formulate the following simplification: during a deformation all middle surface point are displaced in a symmetric way in respect of the meridian plane. In this paper will be analyzed sym-

metrically loaded spherical shell. For this case the differential operator $\mathscr{H}$ of equilibrium equations of spherical shell will be smaller and it presented in Table 2.

<div align="right">Table 2</div>

<div align="center">The operator of equilibrium equations on symmetrical load case</div>

| $M_\rho$ | $M_\varphi$ | $N_\rho$ | $N_\varphi$ |
|---|---|---|---|
| | | $-\dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}$ | $\dfrac{1}{\rho}$ |
| $-\dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}\left(1+\rho\dfrac{\partial}{\partial\rho}\right)$ | $\dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}$ | $-\dfrac{1}{R_0}$ | $-\dfrac{1}{R_0}$ |

where $R_0$ is the radius of curvature of the spherical shell.

The spherical shell is modelled by the circular finite elements $k = 1,2,\ldots,r$, connected in the main discrete model nodes. The functions of the elastic or elastic-plastic spherical shells can have breaks. In order to increase the accuracy of the results, these breaks must be considered. However, the breaks can be modelled only in the nodes, where the elements are connected. Therefore, while choosing the elements mesh, not only the geometry of the structure and distribution of the external loads, but also the possible places of the breaks should be considered.

The spherical shell is investigated in the polar coordinate system $(\rho,\varphi,z)$ with the origin in the center of the structure. It is sufficient to investigate only one radius $(\rho)$ of such a shell, since the internal forces and displacements are independent with respect to the coordinate $\varphi$, when symmetric loading is applied i.e. Table 2. The circular element created by S. Kalanta [6] is used for discretization (Fig. 3).
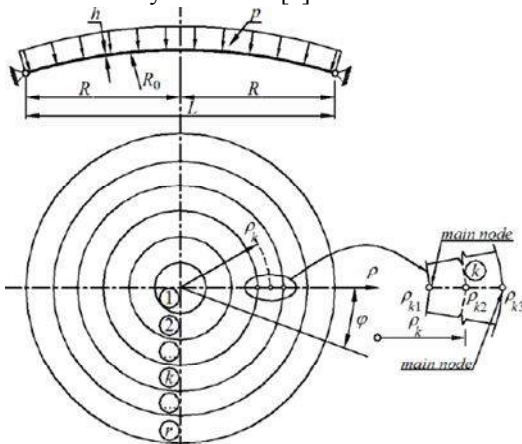


Fig. 3. The discretization of the spherical shell by circular elements with three nodal points

The circular element is investigated in the system of the local coordinates $\varphi$ and $\xi$. The nodal internal forces are shown in Fig. 4. The relation between the global coordinate $\rho_k$ and the local coordinate $\xi_k$ is described by the dependences as follows:

$$\rho_k = \rho_{k2} + 0.5\xi_k b_k , \qquad (3)$$

where $\rho_{k2}$ is the coordinate of the second node in the global coordinate system; $b_k$ is the width of the finite element. The shape functions associated with node $i$, which for the bending moments are described by the quadratic elements with three nodded has form $B(\xi)$ [8]:

$$B_{k1} = \xi_k(\xi_k - 1)/2, \qquad (4a)$$

$$B_{k2} = 1 - \xi_k{}^2, \qquad (4b)$$

$$B_{k3} = \xi_k(\xi_k + 1)/2, \qquad (4c)$$

while the axial forces – by the linear elements with two nodes [8]:

$$B_{k1} = (1 - \xi_k)/2, . \qquad (5a)$$

$$B_{k2} = (1 + \xi_k)/2. \qquad (5b)$$

An external distributed load vector is expressed as follows:

$$\boldsymbol{P}_k = (P_\rho, p_\varphi, p_n)^T \qquad (6a)$$

and for symmetrical loaded

$$\boldsymbol{P}_k = (p_\rho, p_n)^T . \qquad (6b)$$

The internal forces vector $\boldsymbol{S}$ and the external load vector $\boldsymbol{P}$ are related by the differential equations of statics for $k^{th}$ element

$$\mathscr{H}_k \boldsymbol{S}_k = \boldsymbol{P}_k , \quad k = 1, 2, ..., r. \qquad (7)$$

The discrete shell model is regular for circular elements of the same width. The load can be distributed over the surface of the finite elements or concentrated in the main node. The assumption is made that physical properties of the material (the elastic modulus $E$ and Poisson's ratio $v$), shell thickness $h$ and the intensity of the distributed load $p$ in the element are constant.

The stress state of the element is described by the vector of the internal forces $\boldsymbol{S}_k$, which is composed of the internal forces of all three nodes (Fig. 4).
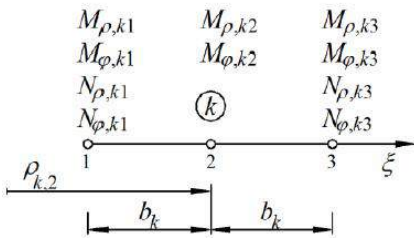
Fig. 4. The internal forces of *kth* element

### 2.1. An equilibrium equations of spherical shell

The shape functions (4)–(5) do not satisfy the differential equilibrium Eqs. (7) of the spherical shell, therefore, it is necessary to write the internal equilibrium equations for every element. Thus, the spherical shell discrete model equilibrium equations are composed not only of the equilibrium equations in the main nodes, where the elements are connected, but also of the internal equilibrium equations of the elements. Algebraic internal equilibrium equations of the finite element are obtained by inserting the functions (4)–(5) into the Eqs. (7) and differentiating using algebraic operator from Table 2:

$$A_k(\xi_k)S_k = p_k, \quad k = 1,2,...,r. \tag{8}$$

The operator of the algebraic equations $A_k(\xi_k)$ is presented in the paper [6–7]. Since the operator $A_k(\xi_k)$ depends on the coordinate $\xi_k$, the static equations of the element can be expressed by the equilibrium equations of the element boundary nodes:

$$A_{n,k}S_k = F_{n,k}, \quad k = 1,2,...,r. \tag{9}$$

Which are created, using Bubnov-Galerkin collocation method. Here

$$A_{n,k} = 2\pi b_k \int_{-1}^{1} G_k(\xi)A_k(\xi_k)S_k(\rho_{k2} + \xi_k b_k)d\xi_k \tag{10}$$

and

$$F_{n,k} = 2\pi b_k \int_{-1}^{1} G_k(\xi_k)A_k(\xi_k)p_k(\rho_{k2} + \xi_k b_k)d\xi_k \tag{11}$$

$$F_{n,k} = \frac{2\pi b_k}{3}\begin{bmatrix} 3p_{k2} - b_k & & & \\ & 3p_{k2} - b_k & & \\ 3p_2 + b_k & & & \\ & & & 3p_{k2} + b_k \end{bmatrix} \times \begin{bmatrix} p_{pk} \\ p_{nk} \end{bmatrix} = \eta_k p_k \tag{12}$$

the matrix of the influence functions $G_k(\xi_k)$ obtained by the differential (4a) and (4c). The element equilibrium equations coefficient matrix $A_{n,k}$ is presented in the paper [6–7]. The static equations of the node $j$, where the elements $k$ and $l$ are connected (Fig. 5), are composed of the equilibrium equations of bending moments, axial and shear forces:



Fig. 5. The internal forces in the main node $j$

$$2\pi\rho_{k3}(-M_{\rho,k3} + M_{\rho,l1}) = 0 \tag{13a}$$

$$2\pi\rho_{k3}(-N_{\rho,k3} - N_{\rho,l1}) = 0 \tag{13b}$$

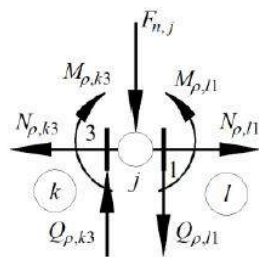$$2\pi\rho_{k3}(Q_{\rho,k3}-Q_{\rho,l1})=2\pi\rho_{k3}F_{n,j}, \tag{13c}$$

where $F_{n,j}$ is the intensity of the normal external load in the main node (in the circular elements connecting joint $j$ of spherical shell).

The static equations of the shear forces are created, using the dependency:

$$Q_{\rho,k}(\xi_k)=\frac{1}{\rho_k}[M_{\rho,k}(\xi_k)-M_{\varphi,k}(\xi_k)+\frac{dM_{\rho,k}(\xi_k)}{d\xi_k}\frac{d\xi_k(\rho_k)}{d\rho_k}. \tag{1.4}$$

The Eqs. (9) of all the elements and the Eqs. (13) of all main nodes make the algebraic system of the equilibrium equations of an elastic shell discrete model:

$$AS_e = F, \tag{1.5}$$

where $A$ $n{\times}m$ size equilibrium equations coefficient matrix of all elements; $S_e$ $m{\times}1$ the discrete model of an elastic internal forces vector of all elements; $F$ $n{\times}1$ size the external load vector of all elements.

**2.2. A geometrical equations of spherical shell**

The geometrical equations of the discrete model are created, applying the principle of virtual forces. With reference to the physical equations $\Theta_e = DS_e$, the compatibility of the displacements $u_e$ and strain $\Theta_e$ in the elastic structure is described by the equation:

$$DS_e - A^T u_e = 0, \tag{16}$$

where $D = diag\mathcal{D}_k$ is the quasi-diagonal flexibility matrix composed of the diagonal blocks, representing the flexibility matrices of the separate finite elements. The flexibility matrix $\mathcal{D}_k$ is presented in the paper [6–7].

The geometrical equations of the elastic-plastic shell are as follows:

$$DS_r + \Theta_p - A^T u_r = 0,. \tag{17}$$

**2.3. A strength conditions of spherical shell**

A von Mises yield criterion for a spherical shell of internal forces is describe in an inequality [7, 9]

$$f_i \equiv N_{0i}^2 - S_i^T \Phi_i S_i \geq 0, \tag{18}$$

where $N_{0i}$ denotes the limit internal forces (the membranous limit axial forces) of the *ith* design section. The matrix

$$\Phi_i = h^2 \times \begin{bmatrix} 16 & -8 & 0 & 0 \\ -8 & 16 & 0 & 0 \\ 0 & 0 & h^{-2} & -h^{-2}/2 \\ 0 & 0 & -h^{-2}/2 & h^{-2} \end{bmatrix} \tag{19}$$

is positive-definite, therefore, the conditions (18) are convex downwards. The condition (18) is homogenous, therefore, it is suitable to apply this condition not only to the analysis and limit equilibrium problems, but also to the optimization problems of the structure's parameters. The admissibility condition of the internal forces for every design section (the element node) $i$ of the discrete model of the structure is based on the non-linear strength condition [10–12] can be rewrite in an inequality

$$\frac{16}{h^2}(M_{\rho i}^2 - M_{\rho i}M_{\varphi i} + M_{\varphi i}^2) + N_{\rho i}^2 - N_{\rho i}N_{\varphi} - N_{\varphi i}^2 \leq (N_{0i})^2. \qquad (20)$$

Assuming that the law of holonomy is valid for the ideal elastic-plastic system, the plastic strains $\boldsymbol{\Theta}_{pi}$ are described by the dependency:

$$\boldsymbol{\Theta}_{pi} = 2\lambda_i \boldsymbol{\Phi}_i \boldsymbol{S}_i, \quad \lambda_i \geq 0, \qquad (21)$$

where $\lambda_i$ is the proportionality strength condition Lagrangian multiplier.

### 3. A mathematical model of the limit load optimization problem

In the limit load optimization problem, it is necessary to determine optimal distribution of the load, corresponding to the given distribution of the limit internal force $N_0$ and the aggregate maximum size load $\boldsymbol{T}^T\boldsymbol{F}_0$, where $\boldsymbol{F}_0$ is the vector of the optimized load parameters. In general, some components of the load can be fixed. Let us suppose that the fixed value load is described by the vector $\boldsymbol{F}_s$, in the equilibrium equations, while the optimized load – by the vector $\boldsymbol{F} = \boldsymbol{\eta}\boldsymbol{F}_0$. Then, the mathematical model of the limit load optimization problem is as follows:

find

$$\max \boldsymbol{T}^T \boldsymbol{F}_0, \qquad (22)$$

$$-\boldsymbol{A}\boldsymbol{S} + \boldsymbol{\eta}\boldsymbol{F}_0 = -\boldsymbol{F}_0, \quad \boldsymbol{F}_0 \geq 0, \qquad (23a)$$

when

$$f_i \equiv N_{0i}^2 - \boldsymbol{S}_i^T \boldsymbol{\Phi}_i \boldsymbol{S}_i \geq 0. \qquad (23b)$$

The solution of this non-linear problem is the vectors $\boldsymbol{S}$ and $\boldsymbol{F}_0$, corresponding to optimal distribution of the optimal ultimate limit load. The residual internal forces are

$$\boldsymbol{S}_r = \boldsymbol{S} - \boldsymbol{S}_e,, \qquad (24)$$

here $\boldsymbol{S}_e$ can be expressed using formulas (15)–(16)

$$\boldsymbol{S}_e = \boldsymbol{D}^{-1}\boldsymbol{A}^T(\boldsymbol{A}\boldsymbol{D}^{-1}\boldsymbol{A}^T)^{-1}(\boldsymbol{\eta}\boldsymbol{F}_0 + \boldsymbol{F}_c),$$

or

$$\boldsymbol{S}_e = \boldsymbol{D}^{-1}\boldsymbol{A}^T\boldsymbol{K}^{-1}(\boldsymbol{\eta}\boldsymbol{F}_0 + \boldsymbol{F}_c), \qquad (25b)$$

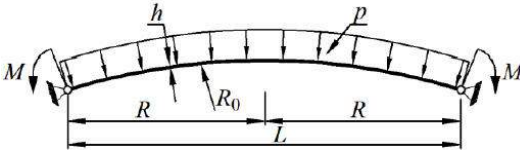where $\boldsymbol{K}$ is the global stiffness matrix of the construction [12–14].

When the number of load optimization parameters $n = 1$, load optimization problem becomes the problem of determining the limit load parameter.

### 4. A numerical research

The purpose of computational research is to test the equilibrium finite element of a spherical shell and to illustrate some research results. A research object was simply supported a shallow spherical shell is shown in Fig. 6. The supports are applied in the outside boundary of shell.

The radius of curvature $R_0=40.0\ m$, the base diameter $R=12.0\ m$, the thickness $h=0.015\ m$. The material – steel: $E=210\ GPa$, $v=0.3$, $f_y = 235\ MPa$. The limit axial force of the shallow spherical shell $N_0 = f_y\ h = 3525\ kN/m$. The out-



Fig. 6. A computational scheme of a spherical shell

side boundary of the shell is loaded by the uniformly distributed moment $M = 5.0\ kNm/m$, and the surface of the shell is subjected to a uniformly distributed load $p$, which is an unknown of the optimization problem. The optimal load distribution is found by using a mathematical model (22)–(23). To analyze the accuracy of the element the 5, 10, 15 and 20 finite element discrete models of the shell are used. A discrete model with 5 elements is presented in Fig. 7. Strength conditions are analyzed in all nodes of the elements.

By using a discrete model with 5 finite elements, the optimal load value $p_5=173.27\ kN/m^2$ have been obtained. Strength conditions become the equations in all the nodes of the 1st and 4th elements and only in the 1st and 3rd nodes of other elements. Plastic deformations are developed in the environment



Fig. 7. A discrete model of the five elements of a spherical shell

of these nodes. The following values of limit axial force were obtained by using 10, 15 and 20 elements discrete models are: $p_{10} = 176.15\ kN/m^2$, $p_{15} = 176.28\ kN/m^2$ and $p_{20} = 176.28\ kN/m^2$. It can be argued that when 5 or 20 element models are taken, the accurate limit load` values have already been obtained, while for 5 elements, the load intensity value $p$ differs
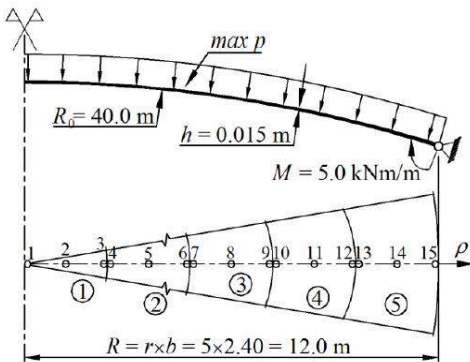
inconsiderably (the error makes 1.02%). The values of the axial forces according 5 elements are given in Table 3.

Table 3

The values of axial forces (*kN/m*)

| Node nr. | 5 elements | | 10 elements | | 15 elements | | 20 elements | |
|---|---|---|---|---|---|---|---|---|
| | $N_\rho$ | $N_\varphi$ | $N_\rho$ | $N_\varphi$ | $N_\rho$ | $N_\varphi$ | $N_\rho$ | $N_\varphi$ |
| 1 | -3390.35 | -3390.35 | -3522.51 | -3522.51 | -3524.80 | -3524.80 | -3524.76 | -3524.76 |
| 2 | -3425.63 | -3460.91 | -3522.85 | -3523.10 | -3524.80 | -3524.83 | -3524.77 | -3524.80 |
| 3 | -3460.91 | -3531.47 | -3523.01 | -3523.34 | -3524.83 | -3524.91 | -3524.81 | -3524.90 |
| 4 | -3460.91 | -3458.32 | -3523.01 | -3523.44 | -3524.83 | -3524.91 | -3524.81 | -3524.91 |
| 5 | -3459.61 | -3455.73 | -3523.22 | -3523.92 | -3524.87 | -3524.99 | -3524.86 | -3524.99 |
| 6 | -3458.32 | -3453.13 | -3523.48 | -3524.48 | -3524.90 | -3525.01 | -3524.89 | -3525.00 |
| 7 | -3458.32 | -3470.67 | -3523.48 | -3524.64 | -3524.90 | -3524.93 | -3524.89 | -3524.92 |
| 8 | -3461.41 | -3476.85 | -3523.77 | -3525.24 | -3524.90 | -3524.79 | -3524.89 | -3524.70 |
| 9 | -3464.49 | -3483.02 | -3524.06 | -3525.86 | -3524.85 | -3524.44 | -3524.81 | -3524.17 |
| 10 | -3464.49 | -3507.45 | -3524.06 | -3524.94 | -3524.85 | -3524.01 | -3524.81 | -3523.73 |
| 11 | -3471.65 | -3521.76 | -3524.21 | -3523.21 | -3524.67 | -3522.83 | -3524.57 | -3521.94 |
| 12 | -3478.81 | -3536.08 | -3524.07 | -3522.90 | -3524.25 | -3520.26 | -3523.95 | -3517.78 |
| 13 | -3478.81 | -3303.09 | -3524.07 | -3520.30 | -3524.25 | -3515.90 | -3523.95 | -3512.95 |
| 14 | -3456.85 | -3259.16 | -3523.59 | -3366.60 | -3522.64 | -3498.17 | -3521.89 | -3478.46 |
| 15 | -3434.88 | -3215.23 | -3489.18 | -3145.01 | -3497.47 | -3133.44 | -3499.94 | -3129.83 |

These values demonstrate high accuracy of the developed finite element in the problems of elastic-plastic state analysis. The values of the bending moments are smaller than axial forces about 1000 times (except the bending moment in the outside boundary of the shell).

**5. Conclusion**

1. An equilibrium finite element is suggested for the analysis of the shallow symmetrically loaded spherical shell. The bending moments and axial forces are described by the quadratic and the linear shape functions.

2. Mathematical models for spherical shell optimization problem of elastic-plastic state analysis for ultimate limit load is formulated (single load). Also, the created element can be effectively used for the elastic-plastic spherical shell optimization problems (for single load or repeated-variable load).

3. The computational analysis performed by using the elements' mesh of various density, has shown high accuracy and convergence of the calculation results. This is particularly important for the analysis of the elastic-plastic shells and for solving the optimization – nonlinear programming problems, whose solution success largely depends on their size (the number of elements).

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