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METHOD OF COMPENSATING LOADS FOR SHALLOW SHELLS. VIBRATION AND STABILITY PROBLEMS

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Based on the integral representation of the displacements functions through Green's functions, the author proposed a method to solve the system of differential equations of the given problem. The equations were solved approximately by reducing to algebraic equations by finite difference techniques in Samarsky scheme. Some examples are given for calculation of eigenvalues of shallow shell vibration problem, which are compared with results received by Onyashvili using Galerkin method.

Introduction

The stability and vibration problems of shallow shells have been studied by many scientists [1], [2]. The usual approaches for those problem were based on the partial differential equations of high order with unknown functions being displacement w and stress Ø functions. Integrating these equations by analytical method usually are too difficult because of the high order of the differential equations even if for bending problems [3].

On the base of the integral representation of displacement functions through Green functions the author has proposed a numerical method for solving the differential equations of the problem. These equations were solved approximately after producing them into linear algebraic equations by finite difference technique.

Governing equations

Vlasov governing differential equations for thin shallow shell with variable curvatures in the form of the three displacements (u,v,w) have been employed [4,5]

$$L_{11}(\overline{u})+L_{12}(\overline{v})+L_{13}(\overline{w})+[(1-\nu^2)/Eh][X_0-m(\partial^2\overline{u})/\partial t^2)]=0\,;$$

$$L_{21}(\overline{u})+L_{22}(\overline{v})+L_{23}(\overline{w})+[(1-\nu^2)/Eh][Y_0-m(\partial^2\overline{v}/\partial t^2)]=0\,;$$

$$L_{31}(\overline{u})+L_{32}(\overline{v})+L_{33}(\overline{w})+[(1-\nu^2)/Eh][Z_0-m(\partial^2\overline{w}/\partial t^2)]=0\,,$$

where $L_{11}, L_{32},\ldots, L_{33}$ – linear differential operator of the shell, h – thickness of the shell, X_0,Y_0,Z_0 – harmonic surface loads located on the shell, m – density of the mass for an unit area, E – Young's modules, ν – Poisson coefficient.

For convenience in integration and computation, the dimensionless *Cartesian* coordinates are used. In the case of free vibration $X_0 = Y_0 = Z_0 = 0$.

The three displacement it he governing equations are assumed in the form

$$\left.\begin{aligned} \bar{u}(X,Y,t) &= u(X,Y)\sin\omega t \\ \bar{v}(X,Y,t) &= v(X,Y)\sin\omega t \\ \bar{w}(X,Y,t) &= w(X,Y)\sin\omega t \end{aligned}\right\}. \tag{1.1}$$

Substituting the above into the governing equations for free vibration of the shells gives

$$\left.\begin{aligned} L_{11}(u)+L_{12}(v)+L_{13}(w) &= \lambda u \\ L_{21}(u)+L_{22}(v)+L_{23}(w) &= \lambda v \\ L_{31}(u)+L_{32}(v)+L_{33}(w) &= \lambda w \end{aligned}\right\}. \tag{1.2}$$

In the case of elastic stability the governing equations of the shell are

$$\left.\begin{aligned} & L_{11}(\bar{u})+L_{12}(\bar{v})+L_{13}(\bar{w}) = 0, \\ & L_{21}(\bar{u})+L_{22}(\bar{v})+L_{23}(\bar{w}) = 0, \\ & L_{31}(\bar{u})+L_{32}(\bar{v})+L_{33}(\bar{w}) = \lambda^* L_{34}(\bar{w}), \end{aligned}\right\} \tag{1.3}$$

where operators in dimensional coordinates are [4,5]

$$L_{11} = \partial^2/\partial X^2 + \left[(1-\nu)/2\right]\left(\partial^2/\partial Y^2\right);\ L_{12} = \left[(1+\nu)/2\right]\left(\partial^2/\partial X\partial Y\right);$$

$$L_{22} = \partial^2/\partial Y^2 + \left[(1-\nu)/2\right]\left(\partial^2/\partial X^2\right);$$

$$L_{13} = -(k_1+\nu k_2)(\partial/\partial X) - k_{12}(1-\nu)(\partial/\partial Y);$$

$$L_{23} = -(k_2+\nu k_1)(\partial/\partial Y) - k_{12}(1-\nu)(\partial/\partial X);$$

$$L_{21} = L_{12};\ L_{31} = L_{13};\ L_{32} = L_{23};$$

$$L_{33} = (D/C)\Delta^2 + k_1^2 + 2\nu k_1 k_2 + k_2^2 + 2(1-\nu)k_{12}^2;$$

with $k_1 = \left(\partial^2 Z / \partial X^2\right);\ k_2 = \partial^2 Z / \partial Y^2;\ k_{12} = \partial^2 Z / \partial X \partial Y$

and $Z = Z(X,Y)$ – the middle surface equation of the shell.

Besides

$$L_{34} = N_x\left(\partial^2/\partial X^2\right) + 2N_{xy}\left(\partial^2/\partial X\partial Y\right) + N_y\left(\partial^2/\partial Y^2\right)\ [4,5];$$

$$\lambda = -m\left[\left(1-\nu^2\right)/EH\right]\omega^2;\ \lambda^* = \left[\left(1-\nu^2\right)/EH\right]N_{cr};$$

$$D = Eh^3/\left[12\left(1-\nu^2\right)\right];\ C = Eh/\left[\left(1-\nu^2\right)\right].$$

Method of analysis

The method to be presented in based on integral representation of the displacement functions through Green functions, by which the governing differential equations of the problem are converted into linear algebraic equations by using finite difference technique.

According to this method, the region of the shell is divided into a set of orthogonal lines $X = X_m\left(m=1,...,M\right)$ and $Y = Y_n\left(n=1,...N\right)$.

The highest derivatives of *u*, *v*, *w* in *Eqs* (1.2) and (1.3) are denoted by:

$$\partial^2 u/\partial X^2 = -k\left(X,Y\right);\ \partial^2 v/\partial X^2 = -s\left(X,Y\right);\ \partial^4 w/\partial X^4 = -p\left(X,Y\right);$$

$$\partial^2 u/\partial Y^2 = -d\left(X,Y\right);\ \partial^2 v/\partial Y^2 = -t\left(X,Y\right);\ \partial^4 w/\partial Y^4 = -q\left(X,Y\right).$$

With the help of integrating along the line $Y = Y_n$, *Eq*. (2.1) can be transformed [6] into

$$\left.\begin{aligned} u &= \int_0^1 f\left(X,\zeta,Y_n\right)k\left(\zeta,Y_n\right)d\zeta; \\ v &= \int_0^1 e\left(X,\zeta,Y_n\right)s\left(\zeta,Y_n\right)d\zeta; \\ w &= \int_0^1 a\left(X,\zeta,Y_n\right)p\left(\zeta,Y_n\right)d\zeta, \end{aligned}\right\} \tag{2.2}$$

where f, e and a are Green functions associated with the (2.1) and the boundary conditions correspond to a clamped shell as follows $u = v = w = w' = 0$ at $X = 0$ and $X = 1$.

The integral equations (2.2) can be reduced to a summation by using Simpson's rule and for the numerical integration and by using second degree interpolation *L* to relate the functions *k*, *s* and *p* at point $\left(\xi, Y_n\right)$ to those at points (X, Y_n). Then *Eqs* (2.2) become

$$\left.\begin{aligned} u_n &= f_n a L_n k_n = F_n \cdot k_n \\ v_n &= f_n a L_n k_n = E_n \cdot k_n \\ w_n &= f_n a L_n k_n = A_n \cdot k_n \end{aligned}\right\}. \tag{2.3}$$

For all the lines paralleled to the $X-$ axis, *Eqs.* (2.3) in matrix notation are $u = Fk,\ \ v = Fk,\ \ w = AP$.

Similarly, *Eqs*. (2.1) can be reduced to

$$u = T^1 HTd^* = H \cdot d\ ,$$

$$v = T^1 GTd^* = G \cdot d\ ,$$

$$w = T^1 BTd^* = B \cdot d \,,$$

where * indicates the sequence of the nodal point along the lines paralleled to X – axis ; T – a unitary transformation matrix to rearrange the nodal points in the Y –direction to the same order as those in the X –direction.

The required derivatives of u,v and w in *Eqs.* (1.2) and (1.3) are obtained by using the derivatives of Green's functions and the procedure of the differential operators. For u, for example, the derivatives are

$$u' = F'k = F'F^{-1}u \,;$$

$$u' = HH^{-1}u \,;$$

$$u'' = -k = -F^{-1}u \,;$$

$$u'' = -d = -H^{-1}u \,;$$

$$u' = FF^{-1}HH^{-1}u \,.$$

In the similar way, the derivatives for v and w can be obtained.

Now we consider the shallow shell for which the middle surface equation is

$$Z(X,Y) = c\left[(X-a)^2 / a^2 + (Y-b)^2 / b^2 - (X-a)^2 (Y-b)^2 / \left(a^2 b^2\right) - 1 \right],$$

By using the dimensionless variables $(X = X/2a)$, $y = Y/2b$ we obtain the differential operators of the shell as follows

$$L'_{ij} = 4a^2 L_{ij}, \quad i, j, = 1, 2, 3, 4.$$

$$L'_{11} = \left(\partial^2/\partial x^2\right) + ((1-v)/2) r^2 \left(\partial^2/\partial y^2\right);$$

$$L'_{12} = ((1+v)/2) r \left(\partial^2/\partial x \partial y\right) = L'_{21};$$

$$L'_{22} = r^2 \left(\partial^2/\partial y^2\right) + ((1-v)/2) r^2 \left(\partial^2/\partial x^2\right);$$

$$L'_{23} = -4r(c/a)\left\{ r^2 \left[1-(2x-1)^2\right] + v\left[1-(2y-1)^2\right]\right\}(\partial/y) +$$
$$+8(c/a) r (1-v)(2x-1)(2y-1)(\partial/\partial x) = L'_{32}\,;$$

$$L'_{13} = -4r(c/a)\left\{ \left[1-(2y-1)^2\right] + vr^2\left[1-(2x-1)^2\right]\right\}(\partial/\partial x) +$$
$$+8(c/a) r (1-v)(2x-1)(2y-1)(\partial/\partial x) = L'_{31}\,;$$

$$L'_{33} = -\left(h^2/48a^2\right)\left[\left(\partial^4/\partial x^4\right) + 2r\left(\partial^4/\partial x^2 \partial y^2\right) + r^4\left(\partial^4/\partial y^4\right)\right] +$$
$$+16(c/a)^2 \left\{\left[1-(2y-1)^2\right]^2 + r^4\left[1-(2x-1)^2\right]^2\right\} +$$

$$+2\nu r^2\left[1-(2y-1)^2\right]\left[1-(2x-1)^2\right]+8r^2(1-\nu)(2x-1)^2(2y-1)^2;$$

$$L'_{34}=\left(N_x/N_{cr}\right)+2r\left(N_{xy}/N_{cr}\right)\left(\partial^2/\partial x\partial y\right)+r^2\left(N_y/N_{cr}\right)\partial^2/\partial y^2;$$

$$\lambda=-4a^2m\left(\left(1-\nu^2\right)/Eh\right)\omega^2;\ \lambda^*=\left(\left(1-\nu^2\right)/Eh\right)N_{cr};\ r=a/b.$$

a. Free vibration problem

Substitution of the derivatives of *u, v* and *w* in *Eqs.* (1.2) and simplification will yield to eigenvalue problem

$$[C-\lambda I]\left\{D^*\right\}=0,$$

where

$$[C]=\begin{Bmatrix} L'_{11}\ L'_{12}\ L'_{13} \\ L'_{21}\ L'_{22}\ L'_{23} \\ L'_{31}\ L'_{32}\ L'_{33} \end{Bmatrix};\quad \left\{D^*\right\}=\begin{Bmatrix} u \\ v \\ w \end{Bmatrix};$$

$$L'_{11}=-F^{-1}-((1-\nu)/2)r^2H^{-1};$$

$$L'_{12}=-4(c/a)\left\{\left[1-(2y-1)^2\right]+\nu r^2\left(1-(2x-1)^2\right)\right\}\ AA^1+$$

$$+8(1-\nu)r^2(c/a)(2x-1)(2y-1)BB^1;$$

$$L'_{21}=\left((1+\nu)/2\right)rF'F^1H^{-1};\ L'_{22}=-r^2G^{-1}-\left((1-\nu)/2\right)E^{-1};$$

$$L'_{23}=-4r(c/c)\left\{r^2\left[1-(2x-1)^2\right]+\nu\left[1-(2y-1)^2\right]\right\}F'F^{-1}+$$

$$+8(c/a)r(1-\nu)(2x-1)(2y-1)AA^1;$$

$$L'_{31}=4(c/a)\left\{r^2\left[1-(2x-1)^2\right]+\nu\left[1-(2y-1)^2\right]\right\}GG^{-1}+$$

$$+8(1-\nu)(c/a)(2y-1)(2y-1)E'E^{-1};$$

$$L'_{33}=-\left(h^2/48a^2\right)\left(-A^{-1}+2r^2A''A'BB^{-1}+r^4B^{-1}\right)+$$

$$+16(c/a)^2\left\{\left[1-(2y-1)^2\right]^2+r^4\left[1-(2x-1)^2\right]\right\}+$$

$$+2\nu r^2\left[1-(2y-1)^2\right]\left[1-(2x-1)^2\right]+8r^2(1-\nu)(2x-1)^2(2y-1)^2.$$

b. The elastic stability problem

In the similar way, *Eqs.* (1.3) can be solved for determining the buckling loads. The differential operators L'_{ij} $(i,j = 1,2,3)$ are the same as formulated in *Eqs.* (2.4), and

$$L'_{34} - \left(N_x / N_{cr}\right) A'' A^{-1} + 2r\left(N_{xy} / N_{cr}\right) A' A^{-1} B B^{-1} + r^2 \left(N_y / N_{cr}\right) B B^{-1}.$$

Substituting L'_{11} ,..., L'_{34} into *Eqs.* (1.3) reduces them to linear algebraic equations:

$$\left[C^* - \lambda^* I\right]\{\overline{w}\} = 0.$$

For non-trivial solution of $\overline{w}$

$$\left[C^* - \lambda^* I\right] = 0,$$

where

$$C^* = -L^{-1}_{34} L'_{31} L'^{-1}_{11} L'_{12} \left(L'_{22} - L'_{21} L^{-1}_{11} L'_{12}\right)^{-1} = \left(L'_{21} L^{-1}_{11} L'_{13} - L'_{23}\right) - L^{-1}_{34} L^{-1}_{11} L'_{13} +$$

$$+ L^{-1} L'_{32} \left(L'_{23} - L'_{21} L^{-1}_{11} L'_{12}\right)^{-1} \left(L'_{21} L^{-1}_{11} L'_{12}\right)^{-1} \left(L'_{21} L^{-1}_{11} L'_{13} - L'_{23}\right) + L^{-1}_{34} L'_{33}.$$

Results and discussions

The free vibration problem was solved for the shallow shell, the middle surface equation of which is

$$Z = c\left[\left(X-a\right)^2 / z^2 + \left(Y-b\right)^2 / b^2 + \left(X-b\right)^2 \left(X-b\right)^2 / \left(a^2 b^2\right) - 1\right],$$

$$X = 2ax; \quad Y = 2by\,.$$

The present results are based on the following dimensions and properties of the shell $a = b = 22{,}8$ *cm*, $h = 0{,}1587$ *cm*, $E = 3{,}3.10^2$ KN/cm^2, $\nu = 0{,}4$. The form of Green function *f, e* and *a* was given by Korenev B.G. [6]

The convergence of the solution for free vibration was shown in Table 1. It is obviously that the convergence is more rapid for low ratio ($c/h = 5$) than for higher ratio $(c/h{=}16)$. It is found that the main factors affecting on the convergence are the mesh size, the rise of thickness ratio, boundary conditions and the degree of Green function used in the solution. It Table 2 the comparison of the results of minimum natural frequency of the shell with Galerkin solution was given.

Table 1

Mesh N x N	R = a/b = 1,0			
	c/h = 5		c/h = 10	
Mode	1st mode	2nd mode	1st mode	2nd mode
3 x 3	28,031	28,031	70,476	70,476
5 x 5	37,333	40,419	69,677	72,204
7 x 7	41,288	41,822	82,608	73,904
9 x 9	40,865	42,171	49,543	81,466
11 x 11	40, 793	41,924	82,998	83,427
13	40,815	42,210	83,526	84,122

Remarks : 1*st* mode – symmetrical with respect to the x and y directions; 2*nd* mode – anti-symmetrical with respect to the x and y directions; Multiplier $\left(1/a^2\right)\sqrt{D/M}$

Table 2

Case	*Method*	ω
c/h = 0	Present method	9,0042
a/b = 1	Galerkin method [2]	9,0359
c/h = 5	Present method	22,536
a/b = 0,5	Galerkin method [2]	26,985
c/h = 5	Present method	40,815
a/b = 1,0	Galerkin method [2]	42,501
c/h = 10	Present method	61,053
a/b = 1,0	Galerkin method [2]	81,294
c/h = 16	Present method	83,426
a/b = 1	Galerkin method [2]	133,255
	Multiplier $\left(1/a^2\right)\sqrt{D/M}$	

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