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SIMULATION OF FORMING A SPATIAL THIN ROD LOCATED IN A CONTINUOUS DEFORMABLE SOLID

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Authors describe the derivation of equations to determine the deformation parameters of the longitudinal axis of a thin rod for a given strain tensor continuum. A single wire of a steel rope is subject to complex deformation when machining, its form and internal stresses are changing as a result. Application of an equations which reflect the most general laws describing form changing of a thin rod requires sophisticated analytical transformations and engaging visual picture of spatial displacements of single points of a rod. It making difficult to conclude calculation formulas. The problem is solved in the framework of a small displacement and deformation hypothesis using tensor analysis. Formulas derived from the new equations coincide with the known results previously obtained on the basis of Clebsch equations and principle of kinematic analogy. On one hand, it confirms the validity of the proposed method and on the other hand it is an additional verification of known formulas. Some examples have been given to illustrate the efficiency of the general equations in tensor form: analyzing the sinusoid forming on a deformable plane, as well as for calculating the deformations of thin helical elements while stretching and twisting helical wire rope. Now there is no need to use a visual picture to describe the displacement of the spatially curved axis of a wire, and all the analysis is carried out by a uniform algorithm. The proposed method to calculate small deformations of a thin rod for a given strain tensor of continuum might be further applied when improving the existing and developing new software for the design of production processes in manufacturing of wire rope and cable.

Keywords: strain tensor, continuous deformable solid, spatial line, thin rod, helical wire rope

1. Introduction

A single wire of a steel rope is subject to complex deformation when machining, its form and internal stresses are changing as a result. During kinematic analysis of such operations one can apply vector equations [1-3] relating the increments of elongation ε and curvature vector $\delta\vec{\Omega}$ with displacement vector $\vec{u}$ and rotation angle $\vec{\varphi}$ of an arbitrary axis of a wire (Clebsch equations)

$$\frac{d\vec{u}}{ds} = \vec{\varphi} \times \vec{\tau} + \varepsilon \cdot \vec{\tau} ,$$

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$$\delta\vec{\Omega} = \frac{d\vec{\varphi}}{ds},$$

where $\vec{\tau}$ - unit tangent vector, s - arc length of the longitudinal axis of a wire.

These equations reflect the most general laws describing form changing of a thin rod. However, their application requires sophisticated analytical transformations and engaging visual picture of spatial displacements of single points of a rod, making it difficult to conclude calculation formulas.

The purpose of this article is to review the known formulas and propose the additional ones for mathematical modelling of changes in the size and shape of a spatial curve which is located in a continuous deformable solid (continuum). The problem is solved in the framework of a small displacement and deformation hypothesis using tensor analysis.

2. General equations to calculate small deformations of a thin rod for a given strain tensor of continuum

Let us consider a continuous deformable solid (continuum), which is rigidly connected to the coordinate system $X_1'X_2'X_3'$ (Fig. 1). Spatial curve is given inside of the body, it is described by parametric equations $X_i'(s)$, where i takes integer values from 1 to 3; s - scalar parameter (are length). Here and below we use the tensor notation of a vector, as well as the rules of tensor summation over repeated index.

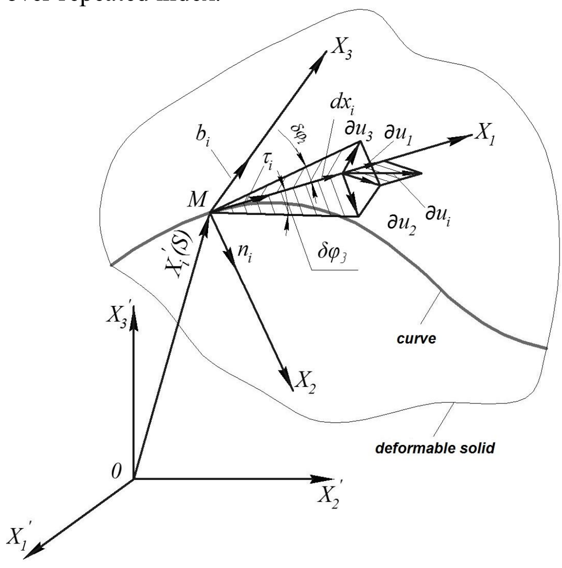

Fig. 1. Deformation scheme of an infinitesimal segment of a spatial line in the vicinity of a given point of a continuum

Assume that in the fixed coordinate system $X_1'X_2'X_3'$ the small deformation tensor of a continuum is set as follows

$$\xi_{ij}' = \frac{\partial u_i'}{\partial x_j'} = e_{ij}' + \omega_{ij}', \tag{1}$$

where u_i' – displacement vector of an arbitrary point of a continuum; e_{ij}' – symmetric tensor of pure deformation; ω_{ij}' – antisymmetric tensor of continuum rotation in the vicinity of a given point [4, 5].

One needs to obtain the formula to calculate the deformation parameters of the spatial line: elongation ε and increment of the curvature vector $\delta\Omega_i$ at an arbitrary point *M*.

Let us introduce a natural coordinate system $X_1X_2X_3$ (Fig. 1) with unit vectors τ_i, n_i, b_i (tangent, normal and binormal to a given curve). The unit vectors of the natural trihedron are determined by the known formulas [5]

$$\tau_i' = \frac{dx_i'}{ds}, \quad n_i' = \frac{1}{\Omega_3}\cdot\frac{d\tau_i'}{ds}, \quad b_i' = \frac{1}{\Omega_1}\cdot\left(\frac{dn_i'}{ds} + \Omega_3\tau_i'\right), \tag{2}$$

where Ω_3, Ω_1 – curvature and rotation angle per unit length of a spatial line.

Having known vector components τ_i', n_i', b_i' projected on a fixed coordinate system one can write down the matrix of the linear transformation A_{ij} to move from the fixed coordinate system $X_1'X_2'X_3'$ to the natural one $X_1X_2X_3$

$$A_{ij} = \begin{pmatrix} \tau_1' & \tau_2' & \tau_3' \\ n_1' & n_2' & n_3' \\ b_1' & b_2' & b_3' \end{pmatrix}. \tag{3}$$

Using the matrix (3) let us determine the strain tensor components in the natural coordinate system

$$\xi_{ij} = A_{ik}A_{jl}\xi_{kl}', \quad e_{ij} = A_{ik}A_{jl}e_{kl}', \quad \omega_{ij} = A_{ik}A_{jl}\omega_{kl}'. \tag{4}$$

Elongation of a given line in the vicinity of an arbitrary point *M* is evidently equal to elongation of a continuum in the axis X_1 direction, i.e. $\varepsilon = e_{11}$. Then taking into account (4) and (3), we obtain the known formula [5]:

$$\varepsilon = A_{1k}A_{1l}e_{kl}' = \tau_k'\tau_l'e_{kl}' = \tau_k'\tau_l'\xi_{kl}'. \tag{5}$$

To analyse the changes in the curvature vector $\delta\Omega_i$ of a given line let us consider the rotation scheme of an infinitesimal tangent vector dx_i in the vicinity of an arbitrary point *M* (Fig. 1). The small angle of rotation around the axis X_1 is due to rotation of continuum as a whole rigid and is determined by tensor ω_{ij} [4, 5]

$$\delta\varphi_1 = -\frac{1}{2}\varepsilon_{1jk}\omega_{jk} = -\frac{1}{2}\left(\varepsilon_{123}\omega_{23} + \varepsilon_{132}\omega_{32}\right) = -\omega_{23} = \omega_{32},$$

where ε_{1jk} – Kronecker symbol.

Taking into account (3) and (4) the last expression can be transformed as follows

$$\delta\varphi_1 = A_{3i}A_{2i}\omega'_{ij} = b'_i n'_j \omega'_{ij}. \quad (6)$$

The small angle of rotation of the vector dx_i around the axis X_2 (Fig. 1) is equal to the partial derivative $\delta\varphi_2 = -\frac{\partial u_3}{\partial x_1}$. According to definition of the total strain tensor the following equality holds $\frac{\partial u_3}{\partial x_1} = \xi_{31}$. Using (3) and (4) one can obtain a formula

$$\delta\varphi_2 = -\xi_{31} = -A_{3i}A_{1j}\xi'_{ij} = -b'_i\tau'_j\xi'_{ij}. \quad (7)$$

Similarly, considering rotation around the axis X_3 on the Fig. 1, one can get one more relationship

$$\delta\varphi_3 = \frac{\partial u_2}{\partial x_1} = \xi_{21} = A_{2i}A_{1j}\xi'_{ij} = n'_i\tau'_j\xi'_{ij}. \quad (8)$$

Then let us analyse the change in curvature and torsion of a rod. Curvature vector of a space curve is determined by the rotation intensity of the natural axis $X_1X_2X_3$ when moving along the arc s. In the original (undeformed) state of the continuum the natural axis rotates by the angle of $d\varphi_{io}$ while moving along the line for the length of ds, so that the vector is equal to the initial curvature $\Omega_{io} = \frac{d\varphi_{io}}{ds}$.

During the deformation the arc length ds of a spatial line increases by the amount of $\delta(ds) = \varepsilon \cdot ds$ whereas rotation angle of the natural trihedron is incremented by $\delta\varphi_i$. Then the new value of the line curvature is equal to

$$\Omega_i = \frac{d\varphi_i}{ds'_i} = \frac{d(\varphi_{io} + \delta\varphi_i)}{ds + \varepsilon \cdot ds} = \frac{\frac{d\varphi_{io}}{ds} + \frac{d(\delta\varphi_i)}{ds}}{1+\varepsilon} = \frac{\Omega_{io} + \frac{d(\delta\varphi_i)}{ds}}{1+\varepsilon}.$$

Multiplying the numerator and denominator of the last equality by $(1-\varepsilon)$ and disregarding the second-order terms in comparison to ε one can obtain

$$\Omega_i = \Omega_{io}(1-\varepsilon) + \frac{d(\delta\varphi_i)}{ds}.$$

Subtract from the last equality initial curvature Ω_{io}, we obtain the formula for calculating the change in curvature vector

$$\delta\Omega_i = \Omega_i - \Omega_{io} = \frac{d(\delta\varphi_i)}{ds} - \varepsilon \cdot \Omega_{io} = \delta\Omega_{i\phi} + \delta\Omega_{i\varepsilon}. \qquad (10)$$

In the last formula the following notations are applied

- the change in vector of a spatial line curvature due to rotation together with the continuum

$$\delta\Omega_{i\varphi} = \frac{d(\delta\varphi_i)}{ds} = \frac{\tilde{d}(\delta\varphi_i)}{ds} + \varepsilon_{ijk}\Omega_j\delta\varphi_k, \qquad (11)$$

where $\frac{\tilde{d}(\delta\varphi_i)}{ds}$ – local derivative of the vector $\delta\varphi_i$; $\varepsilon_{ijk}\Omega_j\delta\varphi_k$ – vector product of Ω_j and $\delta\varphi_k$;

- the change in curvature vector due to strain deformation of the line

$$\delta\Omega_{i\varepsilon} = -\varepsilon \cdot \Omega_{io}. \qquad (12)$$

Set of formulas (2) and (5) - (12) is the general solution of the problem which allows to define all the deformation parameters of a line, located in a continuum.

3. Examples of using the general equations to solve applied problems

Example 1. The curve in the form of a sinusoid $X_1' = p$, $X_2' = \sin p$ (p – a scalar) is in the plane $X_1'X_2'$ (Fig. 2). The deformation of the plane is described by a symmetric tensor (plane strain) $\xi_{ij}' = \begin{pmatrix} \varepsilon_{11} & 0 \\ 0 & \varepsilon_{22} \end{pmatrix}$, then $\omega_{ij}' = 0$ and $e_{ij}' = \xi_{ij}'$.

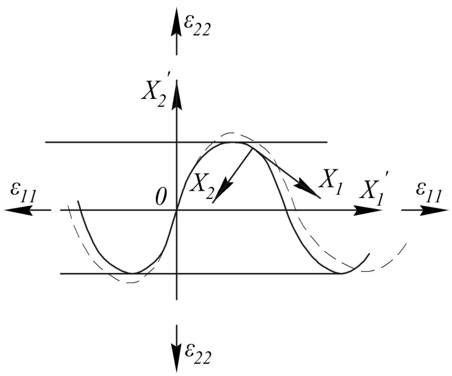

Fig. 2. Plane deformation scheme of a sinusoid

One needs to obtain the formulas for calculating the deformation parameters of a given line.

Sequentially performing mathematical transformations according to (2) and further (5) - (12), in this example one can get the following set of finite relationships:

- for elongation

$$\varepsilon = \frac{\varepsilon_{11} + \cos^2 p \cdot \varepsilon_{22}}{1 + \cos^2 p};$$

- for changing line in torsion and its curvature around the axis X_2

$$\delta\Omega_1 = \delta\Omega_2 = 0;$$

- for changing line curvature around the axis X_3

$$\delta\Omega_{3\varphi} = \frac{sign(\sin p) \cdot \sin^3 p}{(1 + \cos^2 p)^{5/2}}(-\varepsilon_{11} + \varepsilon_{22}),$$

$$\delta\Omega_{3\varepsilon} = \frac{sign(\sin p) \cdot \sin p}{(1 + \cos^2 p)^{5/2}}(-\varepsilon_{11} + \varepsilon_{22} \cdot \cos^2 p).$$

Example 2. Simultaneous tension and torsion of a helical wire rope.

In order to describe geometrical characteristics of a longitudinal axis for an arbitrary wire one applies parametric equations of spiral line [1,3] (Fig. 3):

$$X_1' = s \cdot \cos\alpha,$$
$$X_2' = r \cdot \cos\psi,$$
$$X_3' = r \cdot \sin\psi, \qquad (13)$$

where $\psi = \frac{s}{r} \cdot \sin\alpha$ - angle coordinate of spiral line points.

Substituting (13) in (2) one can obtain the components for unit vectors of natural trihedron as projections on the fixed coordinate system $X_1'X_2'X_3'$

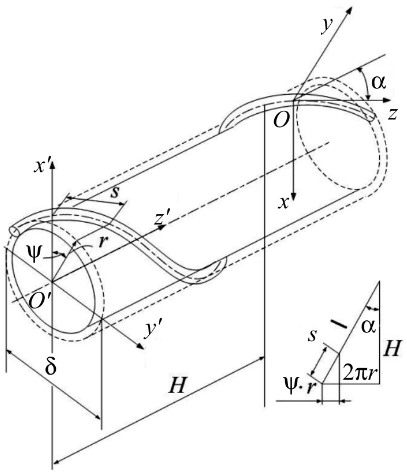

Fig. 3. Coordinate system of a cross section of a wire rope

$$(\tau_i') = (\cos\alpha, -\sin\alpha \cdot \sin\psi, \sin\alpha\cos\psi),$$
$$(n_i') = (0, -\cos\psi, -\sin\psi),$$
$$(b_i') = (\sin\alpha, \cos\alpha \cdot \sin\psi, -\cos\alpha\cos\psi).$$

Let the extensional strain and rotation angle to be set as $\varepsilon' = const$ and $\delta\Omega_1 = const$, respectively. According to [1] we use plain cross section hypothesis, then a wire rope might be considered as a solid cylindrical body with embedded thin spiral wires. It is not difficult to show that in this case components of displacement vector for an arbitrary point of a body are determined by formulas [4]:

$$u_1' = \varepsilon' X_1', \; u_2' = \delta\Omega_1' X_1' X_3', \; u_3' = \delta\Omega_1' X_1' X_2' \qquad (14)$$

Hence according to (1) one can put down deformation tensors of a continuum in the fixed coordinate system

$$\left(\xi_{ij}'\right) = \begin{pmatrix} \varepsilon' & 0 & 0 \\ -\delta\Omega_1' X_3' & 0 & -\delta\Omega_1' X_1' \\ \delta\Omega_1' X_2' & \delta\Omega_1' X_1' & 0 \end{pmatrix},$$

$$\left(e_{ij}'\right) = \begin{pmatrix} \varepsilon' & \dfrac{-\delta\Omega_1' X_3'}{2} & \dfrac{-\delta\Omega_1' X_2'}{2} \\ \dfrac{-\delta\Omega_1' X_3'}{2} & 0 & 0 \\ \dfrac{-\delta\Omega_1' X_2'}{2} & 0 & 0 \end{pmatrix}, \qquad (15)$$

$$\left(\omega_{ij}'\right) = \begin{pmatrix} 0 & \dfrac{\delta\Omega_1' X_3'}{2} & -\delta\Omega_1' X_2' \\ \dfrac{-\delta\Omega_1' X_3'}{2} & 0 & -\delta\Omega_1' X_1' \\ \dfrac{-\delta\Omega_1' X_2'}{2} & -\delta\Omega_1' X_1' & 0 \end{pmatrix}.$$

Then using (5) and taking into account (10) and (15) one gets the elongation of an arbitrary point of a wire spiral axis

$$\varepsilon = \tau_i' \tau_j' e_{ij}' = \cos^2 \alpha \cdot \varepsilon' + r \sin\alpha \cos\alpha \delta\Omega_1'. \qquad (16)$$

Using the expressions (6), (7), (8) it is possible to find rotation angles for an infinitesimal segment of a spiral line

$$\delta\varphi_1 = b_i' n_j' \omega_{ij}' = \cos\alpha X_1' \delta\Omega_1',$$

$$\delta\varphi_2 = -b_i' \tau_j' \xi_{ij}' = -\sin\alpha\cos\alpha \cdot \varepsilon + r\cos^2\alpha\delta\Omega_1', \qquad (17)$$

$$\delta\varphi_3 = n_i' \tau_j' \xi_{ij}' = \sin\alpha X_1' \delta\Omega_1'.$$

Substituting (17) into (10) and disregarding the curvature change due to elongation of an axis we have the following formulas

$$\delta\Omega_1 = \frac{d(\delta\varphi_1)}{ds} = \frac{\sin^3\alpha}{r}\cos\alpha \cdot \varepsilon + \cos^4\alpha \cdot \delta\Omega_1',$$

$$\delta\Omega_2 = \frac{d(\delta\varphi_2)}{ds} = 0, \qquad (18)$$

$$\delta\Omega_3 = \frac{d(\delta\varphi_3)}{ds} = -\frac{\sin^2\alpha\cos^2\alpha}{r}\varepsilon + \sin\alpha\cos\alpha(1+\cos^2\alpha) \cdot \delta\Omega_1'.$$

4. Conclusions

1. Formulas derived from the new equations coincide with the known results previously obtained on the basis of Clebsch equations and principle of kinematic analogy. On one hand, it confirms the validity of the proposed method and on the other hand it is an additional verification of known formulas.

2. The examples mentioned above illustrate the efficiency of the general equations in tensor form – now there is no need to use a visual picture to describe the displacement of the spatially curved axis of a wire, and all the analysis is carried out by a uniform algorithm.

3. The proposed method to calculate small deformations of a thin rod for a given strain tensor of continuum might be further applied when improving the existing and developing new software for the design of production processes in manufacturing of wire rope and cable.

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МОДЕЛЮВАННЯ ФОРМОЗМІНИ ПРОСТОРОВОГО ТОНКОГО СТЕРЖНЯ, ЩО ЗНАХОДИТЬСЯ У СУЦІЛЬНОМУ ДЕФОРМІВНОМУ СЕРЕДОВИЩІ

Розглядається виведення рівнянь для визначення параметрів деформації поздовжньої осі тонкого стержня по заданому тензору деформації суцільного середовища. Наведені приклади використання загальних рівнянь для аналізу формозміни синусоїди на деформівній площині, а також для розрахунку деформації тонких гвинтових елементів при одночасному розтягуванні і крученні спірального канату. Описаний метод дозволяє виключити необхідність використання наочної картини переміщень просторової лінії і побудувати загальний алгоритм аналітичних перетворень для розв'язання прикладних задач.

Ключові слова: тензор деформації, суцільне деформівне середовище, просторова лінія, тонкий стержень, спіральний канат.

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МОДЕЛИРОВАНИЕ ФОРМОИЗМЕНЕНИЯ ПРОСТРАНСТВЕННОГО ТОНКОГО СТЕРЖНЯ, НАХОДЯЩЕГОСЯ В СПЛОШНОЙ ДЕФОРМИРУЕМОЙ СРЕДЕ

Рассматривается вывод уравнений для определения параметров деформации продольной оси тонкого стержня по заданному тензору деформации сплошной среды. Приведены примеры использования общих уравнений для анализа формоизменения синусоиды на деформируемой плоскости, а также для расчета деформации тонких винтовых элементов при одновременном растяжении и кручении спирального каната. Описанный метод позволяет исключить необходимость использования наглядной картины перемещений пространственной линии, и построить общий алгоритм аналитических преобразований для решения прикладных задач.

Ключевые слова: тензор деформации, сплошная деформируемая среда, пространственная линия, тонкий стержень, спиральный канат.

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Хромов В.Г., Хромов І.В., Хромов О.В. **Моделювання формозміни просторового тонкого стрижня, що знаходиться у суцільному деформованому середовищі** // Опір матералів і теорія споруд. – 2016. – Вип. 97. – С. 70 - 78.
Розглядається виведення рівнянь для визначення параметрів деформації поздовжньої осі тонкого стрижня по заданому тензору деформації суцільного середовища.
Табл. 0. Іл. 3. Бібліогр. 5 назв.

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Authors describe the derivation of equations to determine the deformation parameters of the longitudinal axis of a thin rod for a given strain tensor continuum.
Table 0. Fig. 3. Ref. 5.

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Хромов В.Г., Хромов И.В., Хромов О.В. **Моделирование формоизменения пространственного тонкого стержня, находящегося в сплошной деформируемой среде** // Сопротивление материалов и теория сооружений. – 2016. – Вып. 97. – С. 70 - 78.
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