UDC 539.3

LYAPUNOV EXPONENTS ESTIMATION FOR STRONGLY NONLINEAR NONSMOOTH DISCONTINUOUS VIBROIMPACT SYSTEM

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Lyapunov exponents are ones of the most important characteristics for the definition of the dynamical system state. Their estimation for nonsmooth discontinuous system that is vibroimpact system has got certain difficulties. We study their calculation by following the evolution of two nearby orbits in phase space and use three formulas for such estimation. We check this calculation for three different oscillatory regimes: periodic, quasi-periodic and chaotic. We also define the largest Lyapunov exponent by Benettin's algorithm and compare obtained results.
Keywords: nonsmooth dynamic system, Lyapunov exponent, nearby orbits, Benettin's algorithm.

1. Introduction

There is famous book "Chaotic vibrations" by F. Moon [1]. In preface the author writes:

"Had anyone predicted that new discoveries would be made in dynamics three hundred years after publication of Newton's Principia, they would have been thought naïve or foolish. Yet in the last decade new phenomena have been observed in all areas of nonlinear dynamics, principal among these being chaotic vibrations. Chaotic oscillations are the emergence of random like motions from completely deterministic systems".

This discovery essence is that completely determined (deterministic) dynamic system begins to behave by unforeseen chaotic manner when any accidental influence is absent. However, in this unpredictability (chaoticity) it is possible to identify a number of regularities in the system behavior which distinguishes this phenomenon from the classical random processes. Moreover, in contrast to the classical random processes, the phenomena of deterministic chaos can be reproduced in natural, laboratory and numerical experiments. Just deterministic chaos is not an exceptional mode of dynamical systems behavior; on the contrary, such regimes are observed in many dynamical systems in mathematics, physics, chemistry, biology and medicine. Such deterministic chaotic modes are more typical modes than fully predictable (regular) modes. Recently such phenomena are more often described in studies on economics, sociology, philosophy, history. Therefore, the studying of chaotic dynamics is one of the main ways of modern natural science development.

The phenomena of deterministic chaos are possible one only in nonlinear systems.

Therefore the earlier illusions about the possibility of real processes adequate description were dispersed when deterministic chaos was discovered. So the nonlinear dynamics methods are gaining extraordinary weight in modern scientific researches.

Vibroimpact systems (VIS) are strongly nonlinear ones. The right-hand sides of the differential equations describing their movement are discontinuous due the repeated impacts between their elements. Periodic regular regimes can lose stability when system parameters or external influence parameters are changing. Then other modes are arising: quasi-periodic, chaotic, and regimes with great period and with a large number of impacts per cycle – so-called chatter or rattle.

The technique of regular modes studying is well designed; there are a large number of articles and monographs about it (for example, [2-6]). The methodology and the theory of irregular regimes occurrence in nonlinear smooth systems also exist (for example, [7-12]). Also there are some investigations about nonlinear nonsmooth systems (for example, [13-20]. But today there are many unexplored and unknown in such systems.

Vibroimpact systems are exactly such systems - strongly nonlinear, non-smooth, with a discontinuous right-hand side. The investigation of their behavior when both the system parameters and the external influence parameters are changing is of great interest and importance. Vibroimpact systems are just such systems where strong nonlinearity caused by the repeated impacts. Their dynamic behavior studying contains many difficulties because chaotic movements are being implemented in them except regular behavior. It requires an analysis in the field of bifurcation and chaos theory. Bifurcation analysis of smooth systems is sufficiently studied. But the bifurcation analysis of nonsmooth systems with a discontinuous right-hand side in particular mechanical systems with impacts is studied insufficiently. This fact is emphasized by the analysis of world scientific literature.

In addition to the general significance of studying the dynamic behavior of a strongly nonlinear non-smooth vibroimpact system, it is also important to study the dynamic behavior of specific vibroimpact systems. For example, a vibroimpact platform that is widely used in the building industry for concrete mix compaction and concrete products forming [21]. Second example is vibroimpact system depicted at Fig. 1 where the attached body can play the role of percussive or non-percussive dynamic damper.

Thus the chaotic vibrations are the phenomena that are peculiar for nonlinear systems. More than that chaotic vibrations occur when some strong nonlinearity exists. Examples of nonlinearity in mechanical systems include the following:

- nonlinear elastic or spring elements;
- nonlinear damping such as friction;
- backlash, play, or limiters or bilinear springs;
- fluid-related forces;
- nonlinear boundary conditions.

In mechanical continua, nonlinear effects arise from a number of different sources which include the following:

- kinematics; for example, convective acceleration, Coriolis and centripetal accelerations;
- constitutive relations, for example, stress versus strain;
- boundary conditions, for example, free surfaces in fluids, deformation-dependent constraints;
- geometric nonlinearities associated with large deformation in structural solids such as beams, plates and shells.

For nonlinear problems with chaotic dynamics, the time history is sensitive to initial conditions, and precise knowledge of the future may not be possible even when the motion is periodic.

Now system routes to chaos are studying very attentively. But first of all we must know how discern the chaotic vibrations. Sometimes this task is not simple; it is difficult to distinguish such attractor from quasi-periodic or periodic with large period and with great number of impacts per cycle (chatter or rattle). There are some system characteristics that allow distinguishing the chaotic attractor – strange attractor. Ones of such characteristics are Lyapunov exponents in particular the largest Lyapunov exponent.

2. Lyapunov exponents

The tests for chaotic vibrations are qualitative and quantitative and “involve some judgment and experience on the part of the investigator” [1]. Quantitative tests for chaos are available and have been used with some success. One of the most widely used criteria is the Lyapunov exponent.

Chaos in deterministic systems implies a sensitive dependence on initial conditions. The Lyapunov exponent test measures exactly the sensitivity of the system to changes in initial conditions. Conceptually, one imagines a small ball of initial conditions in phase space and looks at its deformation into ellipsoid under the dynamics of the system. If d is the maximum length of the ellipsoid and d_0 the initial size of the initial condition sphere, the Lyapunov exponent λ is interpreted by the equation

$$d(t) = d_0 e^{\lambda(t-t_0)}. \qquad (1)$$

Here we can consider d_0 as a measure of the initial distance between the two starting points, at a small but later time the distance is d .

In other words Lyapunov exponents for dynamic system with continuous time define the degree of distance or rapprochement for different but nearby trajectories at infinity that is the exponentially fast divergence or convergence of nearby orbits in phase space. This means that if two trajectories start close to one another in phase space, they will move exponentially away from each other for small times on the average.

If largest Lyapunov exponent is positive then the distance between initially nearby trajectories is increasing in the course of time. If it is negative then near trajectories is approaching each other some more, if it is zero then near trajectories are staying at the same distance approximately. Let us note that Lyapunov exponents may be different ones for different initial values. So one measurement is not sufficient and the calculation must be averaged over different regions of phase space. This average can be represented by

$$\lambda = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \frac{1}{t_i - t_{0i}} \ln \frac{d_i}{d_{0i}} . \quad (2)$$

Lyapunov exponents may be obtained analytically extremely rarely. There are numerical methods which allow their obtaining with acceptable accuracy.

The calculation of largest Lyapunov exponent is especially important for diagnostic of complicated dynamics regimes. More than that it is enough very often to know the sign of largest Lyapunov exponent – the presence of positive largest exponent is one of the chaos criterion. There are three possibility for such calculation:

- we numerically integrate two copies of dynamic system with nearby initial conditions and follow the distance evolution between them;
- we jointly numerically integrate the main equations and equations in the variations;
- we determine Lyapunov exponents from time series.

The first method may be used when we have got some difficulties with obtaining or numerical solving the equation in variations [22-24]. The second method is the most used. There is famous algorithm by Benettin and all [25, 1, 24] and the special software for its realization. The third method is used when we have not the dynamical equations and must estimate Lyapunov exponents from an experimental time series [26].

We have got some difficulties under obtaining and numerical solving the equations in variations because our mechanical system (Fig. 1) is nonsmooth vibroimpact system with discontinuous right hand side. Therefore we attempted to use the first method.

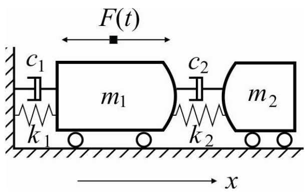

Fig. 1

It is necessary to note that now there are several propositions for Lyaounov exponents calculation for nonsmooth system. The authors of these propositions describe own methods for such estimation [27-38].

3. Lyapunov exponents estimation by following the evolution of two nearby orbits in phase space

We had investigated the dynamic behavior of mechanical vibroimpact system depicted at Fig. 1 in previous studying and had published many results [39]. Now we are studying the system behavior in narrow range of external frequency. We are observing the system route to chaos via destruction of invariant tour that is quasiperiodic regime with two incommensurate frequencies. Therefore we must determine whether oscillatory regime is chaotic one or not. We must estimate the largest Lyapunov exponent for this.

As we said above we'll try to estimate the largest Lyapunov exponent by the first method because we have got some difficulties with obtaining and numerical solving the equation in variations. These difficulties are caused by the nonsmoothness and discontinuity of differential equations right hand sides. We consider the vibroimpact system depicted at Fig. 1. The differential equations of its movement are:

$$\ddot{x}_1 = -2\xi_1\omega_1\dot{x}_1 - \omega_1^2 x_1 - 2\xi_2\omega_2\chi(\dot{x}_1 - \dot{x}_2) - $$
$$-\omega_2^2\chi(x_1 - x_2 + D) + \frac{1}{m_1}[F(t) - F_{con}(x_1 - x_2)],$$

$$\ddot{x}_2 = -2\xi_2\omega_3(\dot{x}_2 - \dot{x}_2) - \omega_1^2(x_2 - x_1 - D) + \frac{1}{m_1}F_{con}(x_1 - x_2), \quad (3)$$

where $\omega_1 = \sqrt{\frac{k_1}{m_1}}$, $\omega_2 = \sqrt{\frac{k_2}{m_2}}$; $\xi_1 = \frac{c_1}{2m_1\omega_1}$, $\xi_2 = \frac{c_2}{2m_2\omega_2}$; $\chi = \frac{m_2}{m_1}$.

External loading is periodic one: $F(t) = P\cos(\omega t + \varphi_0)$, $T = 2\pi/\omega$ is its period.

The term F_{con} is determined by formula (4):

$$F_{con}(z) = K[H(z)z(t)]^{3/2},$$
$$K = \frac{4}{3}\frac{q}{(\delta_1 + \delta_2)\sqrt{A+B}}, \quad (4)$$
$$\delta_1 = \frac{1-\mu_1^2}{E_1\pi}, \quad \delta_2 = \frac{1-\mu_2^2}{E_2\pi},$$

where $z(t)$ is the relative closing in of bodies, $z(t) = x_2 - x_1$, A, B, and q are constants characterizing the local geometry of the contact zone; μ_i and E_i are respectively Poisson's ratios and Young's modulus for both bodies,

The step Heviside function is discontinuous function:

$$H(z)=\begin{cases}1, & z\ge 0\\ 0, & z<0.\end{cases} \quad (5)$$

Therefore its differentiation has got some difficulties. When we use the first method for the largest Lyapunov exponent calculation we integrate the motion equations (3) with two nearby initial conditions. Let us note by the way that we integrate these equations by the program ode23s (MATLAB® ODE solvers). This program integrates the systems stiff differential equations. So we obtain two nearby trajectories. We must follow their distance evolution. It may be fulfilled by the different three formulas. We introduce the following notations.

The coordinates of representing point in phase space for the first and the second trajectories are setting by the vectors $\vec{X}(t),\vec{Y}(t)$ accordingly:

$$\vec{X}(x_1(t),x_2(t),\dot{x}_1(t),\dot{x}_2(t),x_3(t)) . \quad (6)$$

The five coordinate is connected with transition from no autonomous problem to autonomous one and introducing the new variable $x_3=\omega t$.

Then the initial conditions for two trajectories are $\vec{X}(0),\ \vec{Y}(0)$.

Three formulas for largest Lyapunov exponent λ are as follows:

a) We consider one trajectory piece and observe the changing of distance from d_0 to d between two trajectories. Then

$$\lambda\approx\frac{1}{T}\ln\frac{d}{d_0}\text{, where } d=\left\|\vec{X}(T)-\vec{Y}(t)\right\|, d_0=\left\|\vec{X}(0)-\vec{Y}(0)\right\|, \quad (7)$$

T – the trajectory piece length in time.

b) We consider M trajectory pieces of the same time length T and compare the distance between two trajectories at beginning and at the end of each piece. Then

$$\lambda\approx\frac{1}{MT}\sum_{i=1}^{M}\ln\frac{d_i}{d_{i-1}}\text{, where } d_i=\left\|\vec{X}(iT)-\vec{Y}(iT)\right\| . \quad (8)$$

c) We consider M trajectory pieces of the same time length T and compare the distance between two trajectories at the end of each piece with initial distance. Then

$$\lambda\approx\frac{1}{MT}\sum_{i=1}^{M}\ln\frac{d_i}{d_0}\text{, where } d_i=\left\|\vec{X}(iT)-\vec{Y}(iT)\right\|,\ d_0=\left\|\vec{X}(0)-\vec{Y}(0)\right\| . \quad (9)$$

In spite of difficulties under obtaining and numerical solving the equations in variations we have succeeded to use the algorithm of Benettin and all [1, 24, 25] for comparing the obtained results. This algorithm is described in all textbooks, so we will not repeat its description. Note only the following.

When we calculate the largest Lyapunov exponent by Benettin's algorithm we follow the evolution of variations vector $\vec{X}(x_1, x_2, \dot{x}_1, \dot{x}_2, x_3, \tilde{x}_1, \tilde{x}_2, \dot{\tilde{x}}_1, \dot{\tilde{x}}_2, \tilde{x}_3)$.

The largest Lyapunov exponent is given as

$$\lambda \approx \frac{1}{MT}\sum_{i=1}^{M} \ln\|\tilde{x}_i\|. \tag{10}$$

If system of motion differential equations (1) is described in vector form as

$$\dot{x} = X(x), \tag{11}$$

then evolution of small excitement $\tilde{x}(t)$ in linear approach is described by the equation:

$$\dot{\tilde{x}} = \mathbf{A}(t)\tilde{x}. \tag{12}$$

The matrix of equations in variations $\mathbf{A}(t)$ for our vibroimpact system (Fig. 1 and formulas (3), (4)) have got the form:

$$\mathbf{A}(t) = \left(\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right). \tag{13}$$

Here

$a_{31} = -\omega_1^2 - \omega_2^2\chi - \frac{1}{m_1}\frac{\partial F_{con}}{\partial x_1}$, $a_{32} = \omega_2^2\chi - \frac{1}{m_1}\frac{\partial F_{con}}{\partial x_2}$, $a_{33} = -2\xi_1\omega_1 - 2\xi_2\omega_2\chi$,

$a_{34} = 2\xi_2\omega_2$, $a_{35} = -\frac{P}{m_1}\sin(x_3 + \varphi_0)$, $a_{41} = \omega_2^2 + \frac{1}{m_2}\frac{\partial F_{con}}{\partial x_1}$,

$a_{42} = -\omega_2^2 + \frac{1}{m_2}\frac{\partial F_{con}}{\partial x_2}$, $a_{43} = 2\xi_2\omega_2$, $a_{44} = -2\xi_2\omega_2$.

According to formula (2) $F_{con}(x_1 - x_2) = K \cdot (x_1 - x_2)^{\frac{3}{2}} \cdot H(x_1 - x_2)$. Then

$$\begin{aligned} \frac{\partial F_{con}}{\partial x_1} &= K \cdot H(x_1 - x_2) \cdot \frac{3}{2}(x_1 - x_2)^{\frac{1}{2}}, \\ \frac{\partial F_{con}}{\partial x_2} &= -K \cdot H(x_1 - x_2) \cdot \frac{3}{2}(x_1 - x_2)^{\frac{1}{2}}. \end{aligned} \tag{14}$$

Since this really is a periodically driven oscillator, changes of distances in the phase space direction $x_3 = \omega \cdot t$ are zero, as manifested by the row of zeroes in the matrix **A** (13).Thus to find the largest Lyapunov exponent in this problem one can work in the projection of the phase space $(x_1, x_2, \dot{x}_1, \dot{x}_2, x_3)$ onto the phase plane $(x_1, x_2, \dot{x}_1, \dot{x}_2)$, using the inner bracketed matrix in (13).

We see (5) that step Heviside function $H(x_1 - x_2)$ is discontinuous one. Therefore under integration both initial equations (3) and equations in variations (12,13) we must take into attention zero and nonzero for this function.

4. Numerical results

We have obtained the amplitude frequency response for vibroimpact system (Fig.1 and formulas (3), (4)) before [39,40]. Now we investigate the narrow frequency range and estimate the largest Lyapunov exponent for three oscillatory regimes – periodic, quasiperiodic and chaotic. Underline once more that it depends on initial conditions and may be found only after averaging of several results. We make conclusion about regime kind after the sign of Lyapunov exponent.

a) Periodic regime with *T*-period and one impact per cycle ($\omega = 7{,}40\, rad \cdot s^{-1}$).

The phase trajectories are attracted to closed curve, Poincare map – to one point. This regime is depicted at Fig.2. The initial points are shown at this Figure.

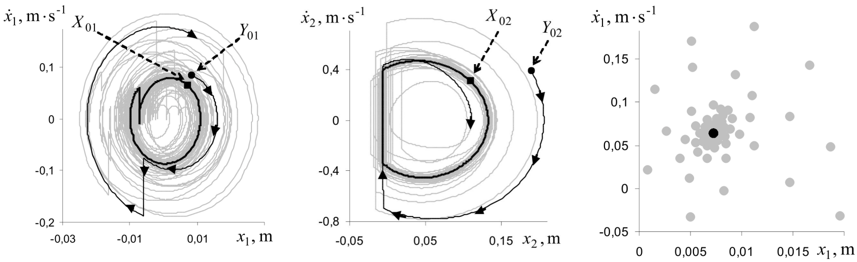

Fig. 2

We represent obtained results in Table 1.

Table 1

Number of intervals, *M*	Formula (7)	Formula (8)	Formula (9)	Benettin's algorithm (10)
100	-0,058	-0,058	-2,35	-0,038
500	-0,031	-0,043	-13,0	-0,061
1000	-0,017	-0,023	-17,4	-0,064
5000	-0,0034	-0,0038	-19,7	-0,066
10000	-0,015	-0,0017	-19,0	-0,067

The number of intervals M given in first column is used for Formulas (8),(9) and Benettin's algorithm. The interval length is the same in this formulas, it is $T = 2\pi/\omega$. In Formula (7) there is only one interval, its length is $T = M \cdot 2\pi/\omega$. This definition is the same one for quasiperiodic and chaotic regimes.

For periodic regime we can estimate the largest Lyapunov exponent by Floquet multiplier value as $\lambda = \frac{1}{T}\ln|\rho|$. This value is $\lambda = -0{,}056$.

What does speak this Table about?

Firstly we see rather big difference in Lyapunov exponent values in different Table cells. The strong dependence on initial conditions and calculating formulas explains this fact.

Secondly we see much unexpected result under calculation by Formula (9). This fact has got such explanation. The phase trajectories for periodic regime are attracted to closed curve therefore they become intimate to this curve shorter and shorter with time. So the distance d_i between two neighboring trajectories becomes smaller and smaller. Then $\ln\frac{d_i}{d_0}$ becomes smaller and smaller too.

Thirdly all obtained Lyapunov exponents are negative ones. Exactly the sign of the largest Lyapunov exponent is important one because we make conclusion about the kind of oscillatory regime via this sign.

We can conclude that Lyapunov exponents estimation by following the evolution of two nearby orbits in phase space gives good result for periodic regime. Let us see how this method works for quasiperiodic regime.

b) Quasiperiodic regime ($\omega = 7{,}46\, rad \cdot s^{-1}$).

This regime is depicted at Fig. 3. It is shown the phase trajectories and Poincare sections at this Figure.

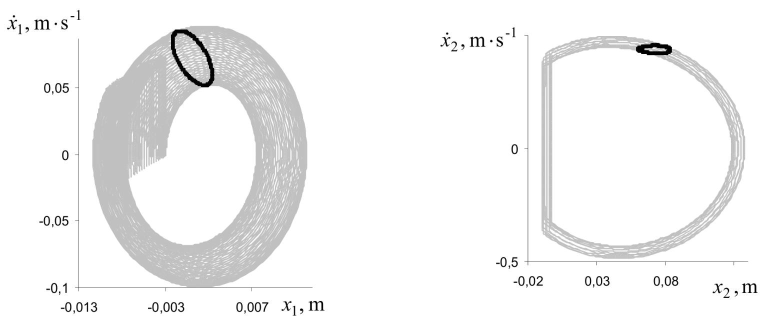

Fig. 3

We represent obtained results in Table 2. The calculations by Benettin's algorithm were made twice for two different initial conditions (I and II).

Table 2

Number of intervals, *M*	Formula (7)	Formula (8)	Formula (9)	Benettin's algorithm(10)	
				I	II
100		0,000243	0,024	0,022	0,019
500		3,81E-05	-0,015	-0,0050	-0,0037
1000		0,000243	-0,026	-0,0069	-0,0067
5000		3,46E-05	-0,035	-0,0089	-0,0089
10000	0,000243	2,76E-05	-0,036	-0,0091	-0,0091

The largest Lyapunov exponent for quasiperiodic regime must be equal to zero. Formula (9) gives the most bad result.

c) Chaotic regime ($\omega = 7{,}92 rad \cdot s^{-1}$).

This regime is depicted at Fig.4. It is shown the phase trajectories and Poincare sections at this Figure.

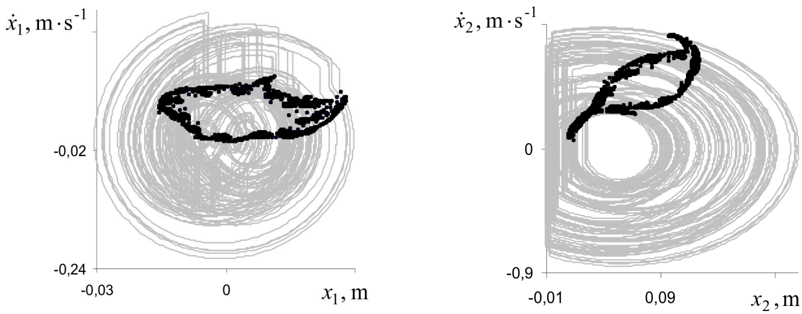

Fig. 4

We represent obtained results in Table 3. The calculations by Benettin's algorithm were made twice for two different initial conditions.

Table 3

Number of intervals, *M*	Formula (7)	Formula (8)	Formula (9)	Benettin's algorithm(10)	
				I	II
100		0,0106	-1,16	0,012	0,069
500		-0,0007	-0,73	0,0094	0,027
1000		8,6E-05	-0,70	0,014	0,018
5000		0,00016	-0,84	0,0060	0,014
10000	0,00018	0,00019	-1,64	0,0030	0,014

We see again the difference in results and bad values after Formula 9. It is importantly that the sign of the largest Lyapunov exponent is positive. Exactly positive sign characterizes the chaotic regime. The largest Lyapunov exponents estimated by Benettin's algorithm for two different initial conditions (two right columns in Table 3) are depicted at Fig. 5,a,b for clearness.

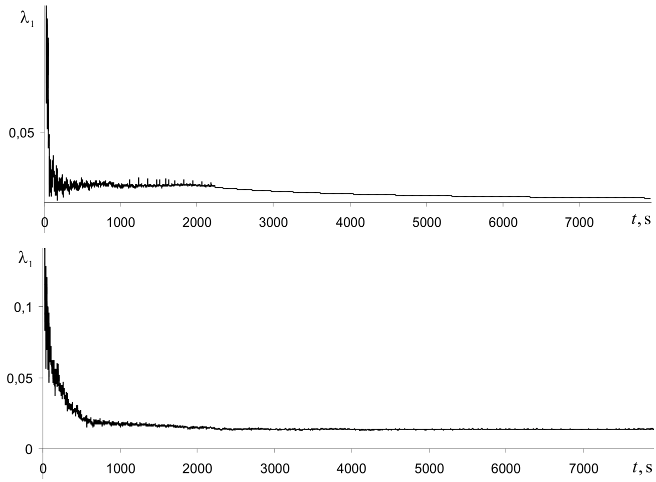

Fig. 5

The comparison of the largest Lyapunov exponents by Benettin's algorithm for different oscillatory regimes is shown at Fig. 6.

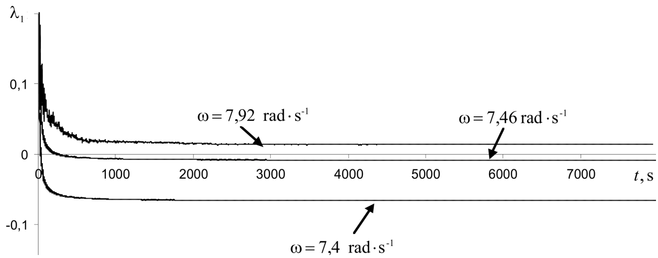

Fig. 6

3. Conclusions

- Lyapunov exponents estimation by following the evolution of two nearby orbits in phase space allows to obtain their values roughly and to determine their signs.

- The calculations after Formula 7 and 8 are preferable. Formula 9 is recommended in several textbooks. None the less it gives worse result and we don't recommend its using.
- We have succeeded to estimate the largest Lyapunov exponent for strongly nonlinear nonsmooth discontinuous vibroimpact system after Benettin's algorithm and have obtained well sufficiently reliable results.

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ОЦІНКА ЛЯПУНОВСЬКИХ ХАРАКТЕРИСТИЧНИХ ПОКАЗНИКІВ ДЛЯ СИЛЬНО НЕЛІНІЙНОЇ НЕГЛАДКОЇ РОЗРИВНОЇ ВІБРОУДАРНОЇ СИСТЕМИ

Ляпуновські показники є одними з найважливих характеристик , що необхідні для визначення стану динамічної системи. Їхня оцінка для негладкої розривної системи, якою і є віброударна система, викликає певні складності. В статті вивчається їхнє обчислення шляхом слідкування за характером еволюції відстані між зображуючи ми точками у часі для двох копій динамічної системи з близькими початковими умовами. Для обчислення використовуються три різних формули. Оцінка провіряється для трьох різних коливальних режимів: періодичного, квазіперіодичного та хаотичного. Вдалося також знайти старший Ляпуновський показник за допомогою відомого алгоритму Бенеттіна та порівняти результати обчислень усіма цими способами.

Ключові слова: негладка динамічна система, показник Ляпунова, близькі орбіти, алгоритм Бенеттіна.

Баженов В.А., Погорелова О.С., Постникова Т.Г.

ВЫЧИСЛЕНИЕ ЛЯПУНОВСКОГО ПОКАЗАТЕЛЯ ДЛЯ СИЛЬНО НЕЛИНЕЙНОЙ НЕГЛАДКОЙ РАЗРЫВНОЙ ВИБРОУДАРНОЙ СИСТЕМЫ

Ляпуновские показатели – это одни из важнейших характеристик, необходимых для определения состояния динамической системы. Их оценка для негладкой разрывной системы, каковой и является виброударная система, представляет определенные трудности. В статье изучается их вычисление путем отслеживания характера эволюции расстояния между изображающими точками во времени для двух копий динамической системы с близкими начальными условиями. При этом для вычисления используются три различных формулы. Оценка проверяется для трех различных колебательных режимов: периодического, квазипериодического и хаотического. Также удалось определить старший Ляпуновский показатель с помощью известного алгоритма Бенеттина и сравнить результаты вычислений всеми этими способами.

Ключевые слова: негладкая динамическая система, показатель Ляпунова, близкие орбиты, алгоритм Бенеттина.

УДК 539.3

Баженов В.А., Погорелова О.С., Постнікова Т.Г. **Оцінка ляпуновських характеристичних показників для сильно нелінійної негладкої розривної віброударної системи**// Опір матеріалів і теорія споруд. – 2017. – Вип. 99. – С. 90 – 105.

Ляпуновські показники є одними з найважливих характеристик , що необхідні для визначення стану динамічної системи. Їхня оцінка для негладкої розривної системи, якою і є віброударна система, викликає певні складності. В статті вивчається їхнє обчислення шляхом слідкування за характером еволюції відстані між зображуючи ми точками у часі для двох копій динамічної системи з близькими початковими умовами. Для обчислення використовуються три різних формули. Оцінка провіряється для трьох різних коливальних режимів: періодичного, квазіперіодичного та хаотичного. Вдалося також знайти старший Ляпуновський показник за допомогою відомого алгоритму Бенеттіна та порівняти результати обчислень усіма цими способами

UDC 539.3

Bazhenov V.A., Pogorelova O.S., Postnikova T.G. **Lyapunov exponents estimation for strongly nonlinear nonsmooth discontinuous vibroimpact system** // Strength of Materials and Theory of Structures. – 2017. – Issue. 99. – P. 90 – 105.

Lyapunov exponents are ones of the most important characteristics for the definition of the dynamical system state. Their estimation for nonsmooth discontinuous system that is vibroimpact system has got certain difficulties. We study their calculation by following the evolution of two nearby orbits in phase space and use three formulas for such estimation. We check this calculation for three different oscillatory regimes: periodic, quasi-periodic and chaotic. We also define the largest Lyapunov exponent by Benettin's algorithm and compare obtained results.

Table 3. Fig. 6. Ref. 37

УДК 539.3

Баженов В.А., Погорелова О.С., Постникова Т.Г. **Вычисление ляпуновского показателя для сильно нелинейной негладкой разрывной виброударной системы** // Сопротивление материалов и теория сооружений. – 2017. – Вып. 99. – С. 90 – 105.

Ляпуновские показатели – это одни из важнейших характеристик, необходимых для определения состояния динамической системы. Их оценка для негладкой разрывной системы, каковой и является виброударная система, представляет определенные трудности. В статье изучается их вычисление путем отслеживания характера эволюции расстояния между изображающими точками во времени для двух копий динамической системы с близкими начальными условиями. При этом для вычисления используются три различных формулы. Оценка проверяется для трех различных колебательных режимов: периодического, квазипериодического и хаотического. Также удалось определить старший Ляпуновский показатель с помощью известного алгоритма Бенеттина и сравнить результаты вычислений всеми этими способами.

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